1. Probabilistic Models

1.1 Learning Objectives

By the end of this lecture, you will be able to:

- Understand what constitutes a probabilistic model
- State and interpret Kolmogorov's three axioms
- Apply the discrete probability law and uniform probability law
- Derive basic properties from the axioms

1.2 Building a Probabilistic Model

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To be a proper probability model, the probability law must follow certain rules — the **axioms**.

1.3 Probability Laws

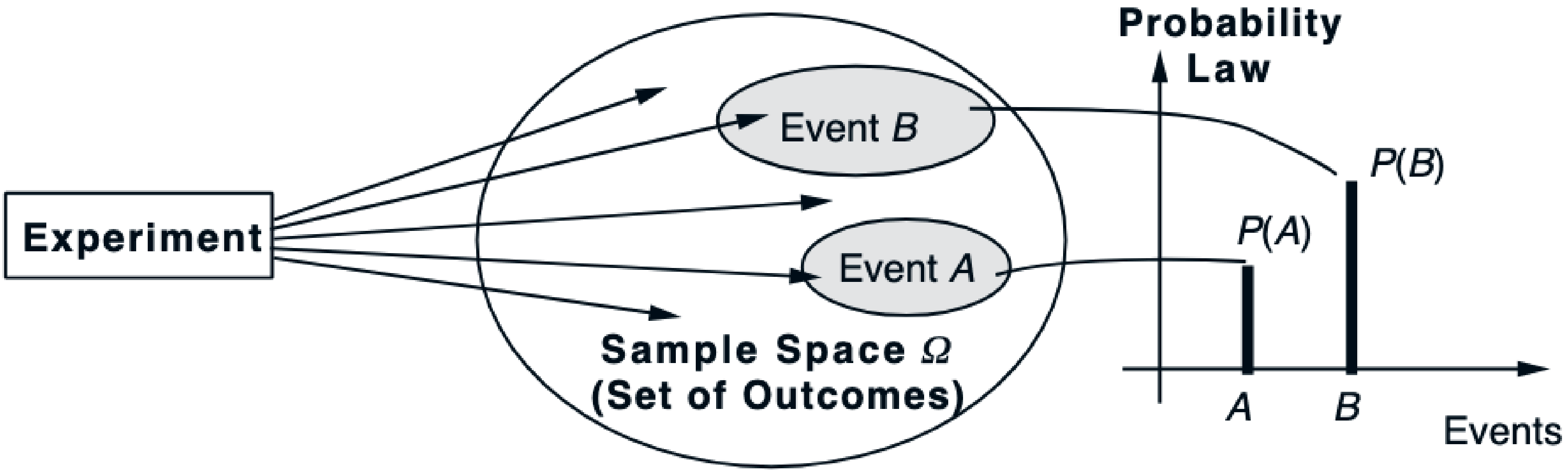

Figure 1.2: The main ingredients of a probabilistic model.

Bertsekas and Tsitsiklis

1.4 Probability Laws

We’ve already seen a certain kind of probability law…

Can you describe the law that governs the probability of outcomes for a fair die?

1.5 Kolmogorov’s Axioms

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Definition: Axiom 3 (Additivity)

If A and B are disjoint events, then $P(A \cup B) = P(A) + P(B)$ More generally, for disjoint $A_1, A_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

1.6 Interpreting the Axioms: The Mass Analogy

Think of probability as **spreading one unit of mass** over the sample space.

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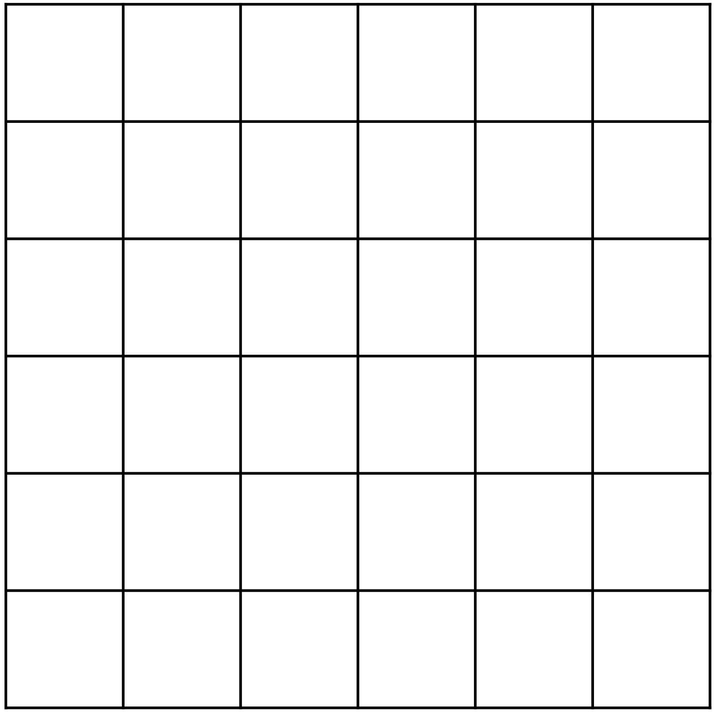
$P(A)$ = total mass assigned to outcomes in A

1.7 Mass Analogy

Draw pile of sand spread over sample space



1.8 Properties from the Axioms

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Example: Empty Set Probability

We know: $S \cup \emptyset = S$ and they're disjoint

By additivity: $P(S) + P(\emptyset) = P(S)$

Therefore: $P(\emptyset) = 0$

1.9 The Complement Rule

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Note: This is often easier than computing $P(A)$ directly!

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1.11 The Union Bound

Union Bound: $P(A \cup B) \leq P(A) + P(B)$

(Equality when A and B are disjoint)

1.12 Inclusion-Exclusion Principle

Definition: Inclusion-Exclusion (Two Events)

For any events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Draw Venn diagram showing A, B, and $A \cap B$

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Note: We'll explore inclusion-exclusion for more than two events in a later lecture.

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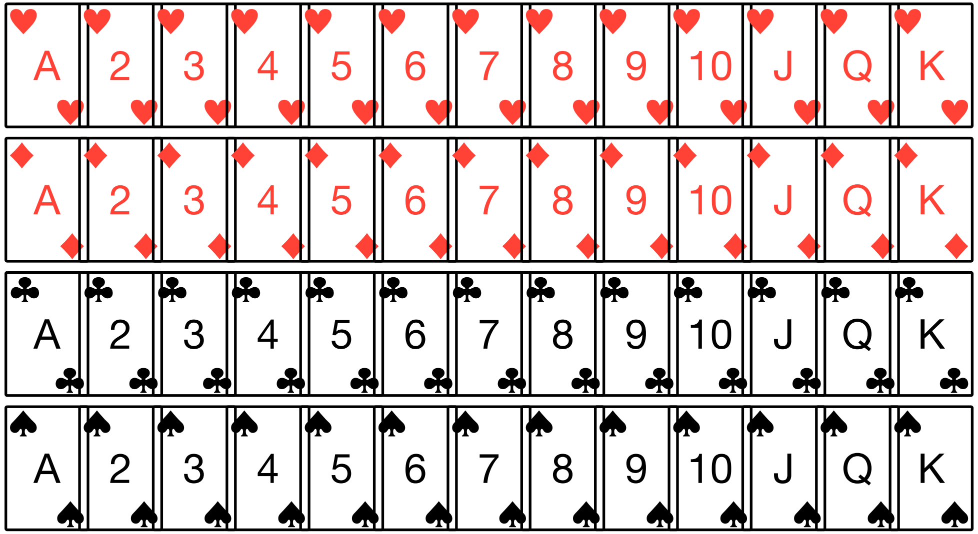

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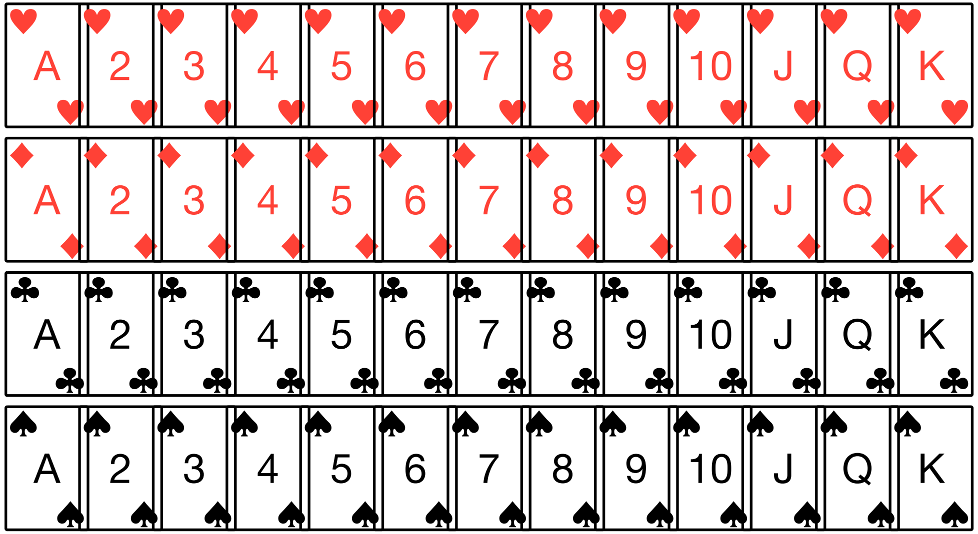

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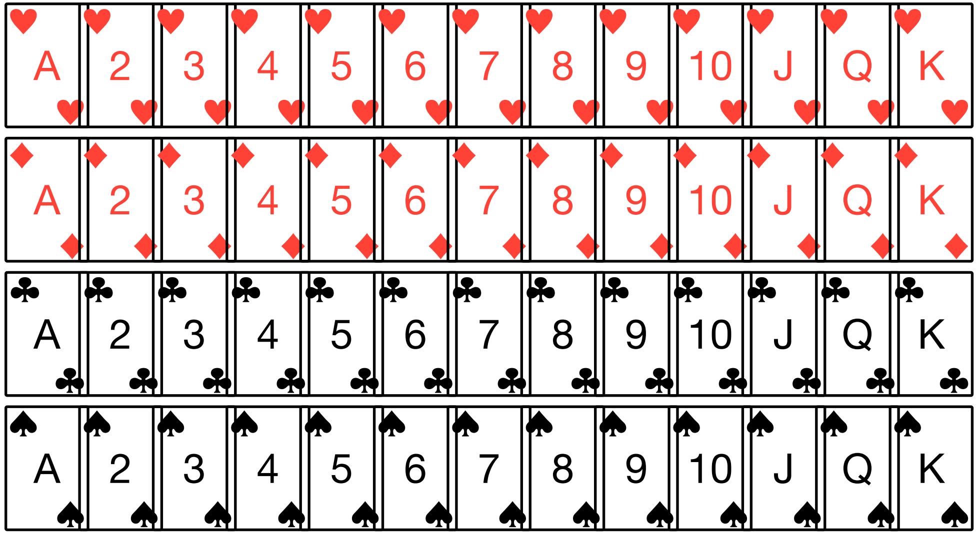

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$P(\text{hearts and face card})$

$P(\text{hearts or face card}) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}$

1.14 Inclusion-Exclusion Principle (Three Events)

For any events A, B, and C:

$$P(A \cup B \cup C) =$$

$$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

1.15 Inclusion-Exclusion Principle (General Case)

For any events $A_1, A_2, ..., A_n$:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - ... + (-1)^{n+1} P(A_1 \cap A_2 \cap ... \cap A_n)$$

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Note: This generalizes: any union can be written as a disjoint union by “removing” previous sets.

2. Discrete Probability Laws

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Example: Two Network Games

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For Game 1: $S = \{0, 1, 2, ..., 10\}$ (total count of intact packets)

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Note: The sample space should capture exactly what matters for the problem.

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Requirements (from axioms):

- Each $p_i \geq 0$ (non-negativity)
- $\sum_{i=1}^{n} p_i = 1$ (normalization)

2.3 Discrete Probability Law

Definition: Discrete Probability Law

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This follows directly from the additivity axiom!

2.4 Special Case: Equally Likely Outcomes

Definition: Discrete Uniform Probability Law

If all n outcomes are equally likely:

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This is exactly our **naive definition** from week 1!

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- $P(\{HHH\}) = 0.6 \times 0.6 \times 0.6 = 0.216$
- $P(\{HHT\}) = 0.6 \times 0.6 \times 0.4 = 0.144$
- $P(\{TTT\}) = 0.4 \times 0.4 \times 0.4 = 0.064$
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Solution: Use a **probability density function** (PDF) $f(x)$ and integrate:

$$P(A) = \int_A f(x)\,\mathrm{d}x$$

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- $P(2.9 < T < 3.1) > 0$

Note: We can only assign positive probability to **intervals** (or unions of intervals).

3.3 Continuous Uniform Distribution

Definition: Continuous Uniform Law

If outcomes are “equally likely” over interval $[a, b]$:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

For any subinterval $[c, d] \subseteq [a, b]$:

$$P(c \leq X \leq d) = \frac{d-c}{b-a} = \frac{\text{length of event}}{\text{length of sample space}}$$

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$$P(0.3 \leq X \leq 0.7) = 0.7 - 0.3 = 0.4$$

$$P(X = 0.5) = 0$$

$$P(X \leq 0.25 \text{ or } X \geq 0.75) = 0.25 + 0.25 = 0.5$$

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$$P(\text{meet}) = 1 - 2 \cdot \frac{(0.75)^2}{2} = 1 - \frac{9}{16} = \frac{7}{16}$$

Draw unit square with shaded meeting region