1. Conditional Probability

1.1 Learning Objectives

By the end of this lecture, you will be able to:

- Define and compute conditional probabilities
- Apply the multiplication rule
- Understand how conditioning reduces the sample space
- Use conditional probability in sequential experiments

1.2 Motivating Example

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Now suppose someone tells you: "A die has been rolled and the result was even."

What's the probability of a 6 **given this information**?

$$S = \{\cancel{1}\ \textcircled{2}\ \cancel{3}\ \textcircled{4}\ \cancel{5}\ \textcircled{6}\}$$

$$P(6 \mid \text{even}) = \frac{1}{3}$$

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- We rescale probabilities so they sum to 1 over B

$S = \{1, 2, 3, 4, 5, 6\}$

$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

$S' = B = \{2, 4, 6\}$

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

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Definition: Conditional Probability

The conditional probability of A given B is:

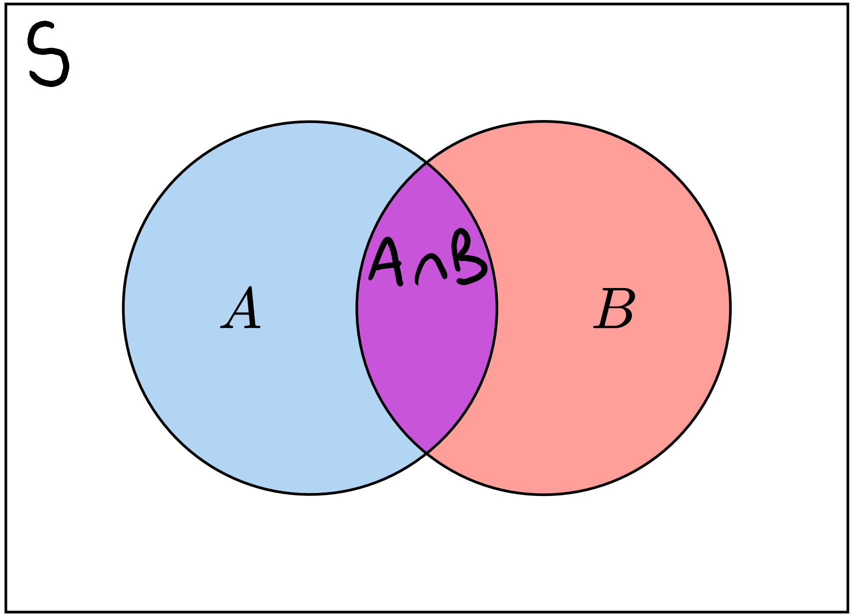
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.

$$P(G \mid \underline{\text{negative}}) \quad P(\text{negative}) = 0$$

$= \text{undef.}$

1.4 Visualizing Conditional Probability



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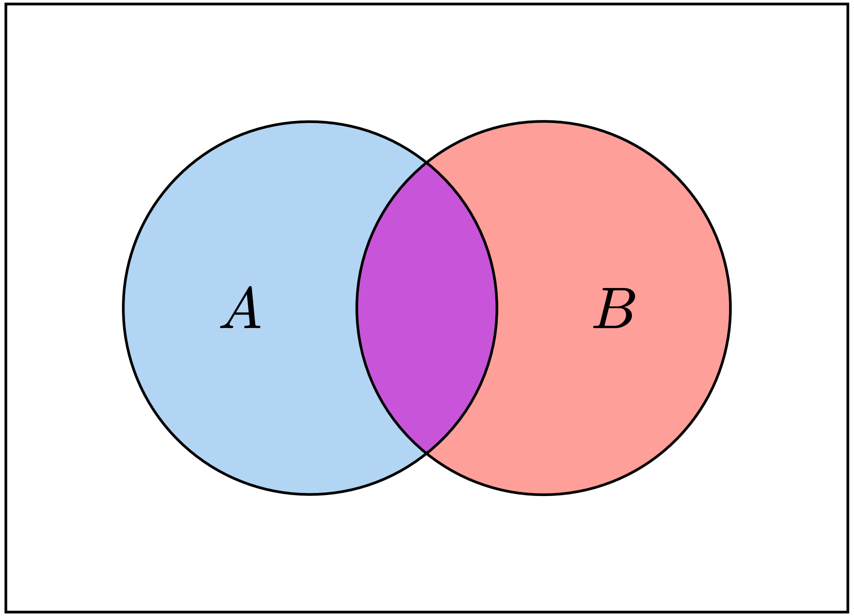

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- Purple region is $A \cap B$

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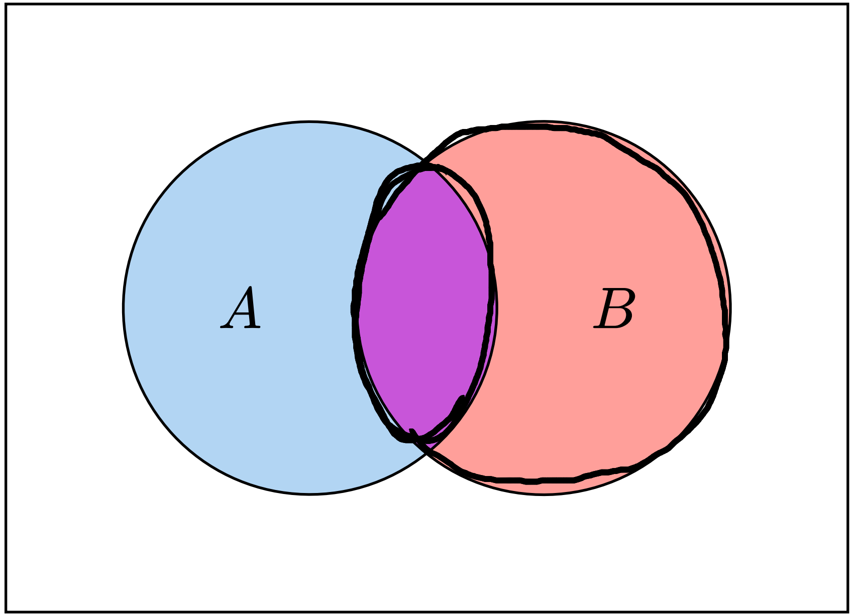

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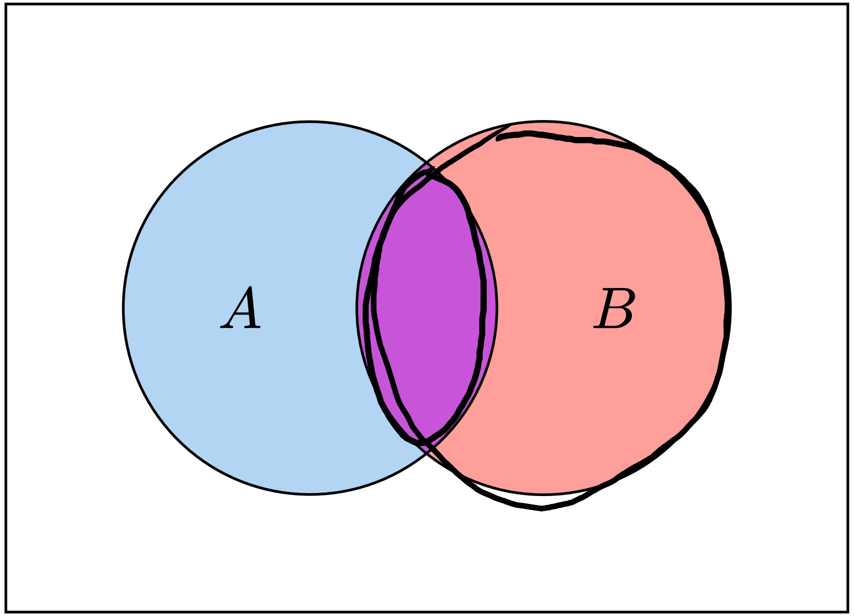

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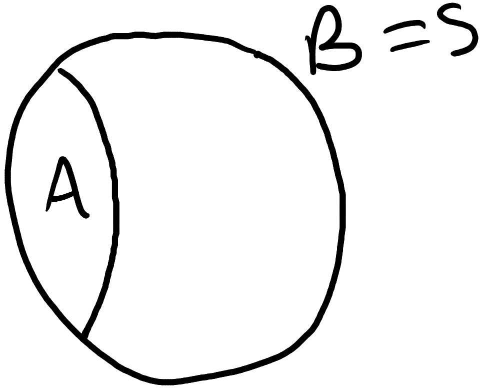

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- If B doesn’t occur on a line: write **“NA”** (not applicable)
- If B does occur on a line: write **“Yes”** if A also occurs, **“No”** otherwise

A	B	A | B	
Yes	Yes	Yes	2
No	Yes	No	
Yes	No	N/A	
No	No	N/A	

A ∩ B is true

$$P(A \mid B) = \frac{1}{2}$$

B true

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Note: We only look at lines where B occurred, then ask: among those, what fraction also has A?

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1.6 Notebook Example: Two Dice

Roll two dice. Let X = first die, S = sum of both dice.

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$P(X=1 \mid S=6)$

Line	X	S	X = 1 and S = 6	X = 1 | S = 6
1	3	8	No	NA
2	1	6	Yes	Yes
3	2	7	No	NA
4	5	6	No	No
5	1	4	No	NA
6	4	6	No	No

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2	1	6	Yes	Yes
3	2	7	No	NA
4	5	6	No	No
5	1	4	No	NA
6	4	6	No	No

- $P(X = 1 \text{ and } S = 6) = \frac{1}{36}$ (1 Yes out of 6 lines total)
- $P(X = 1 \mid S = 6) = \boxed{\frac{1}{3}}$ (1 Yes out of 3 non-NA lines)

1.7 Example: Drawing Cards

Draw one card from a standard deck. Let:

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13 heart cards

What is $P(A \mid B)$?

$$\frac{1}{13}$$

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What is $P(A \mid B)$?

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/\cancel{52}}{13/\cancel{52}} = \boxed{\frac{1}{13}}$$

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What is $P(A \mid B)$?

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/52}{13/52} = \frac{1}{13}$$

There are 13 hearts, and exactly 1 is an Ace.

1.8 The Multiplication Rule

Rearranging the definition:

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$

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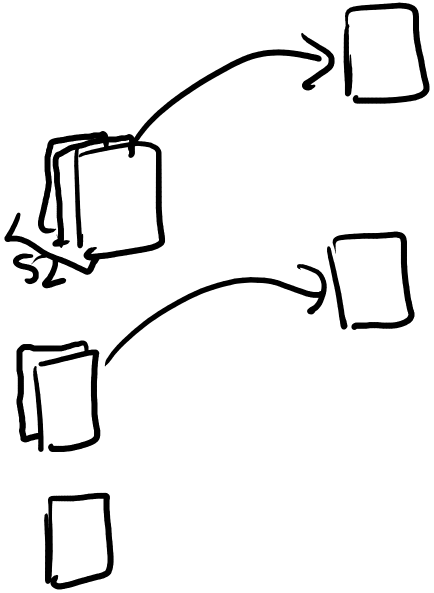
Note: Very useful for sequential experiments!

1.9 Example: Drawing Without Replacement

Draw 2 cards from a deck **without replacement.**

What's $P(\text{both are Aces})$?



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$$P(A_1 \cap A_2) = \underline{P(A_1)} \cdot P(A_2 \mid A_1)$$

$$= \frac{\boxed{4}}{\boxed{52}} \cdot \frac{\boxed{3}}{\boxed{51}}$$

$$= \frac{12}{2652} = \frac{1}{221}$$

$$\frac{4}{52} \cdot \frac{4}{52}$$

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$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1) \cdot P(A_2 \mid A_1) \\ &= \frac{4}{52} \cdot \frac{3}{51} \\ &= \frac{12}{2652} = \frac{1}{221} \end{aligned}$$

After drawing an Ace, only 3 Aces remain among 51 cards.

2. Tree-Based Modeling

2.1 Example: Radar Detection

A radar system monitors airspace for aircraft.

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Given information:

- An aircraft is present with probability 0.05
- If aircraft present: radar detects it with probability 0.99
- If no aircraft: radar falsely registers presence with probability 0.10

classifier

C = { aircraft detected,
aircraft not detected }

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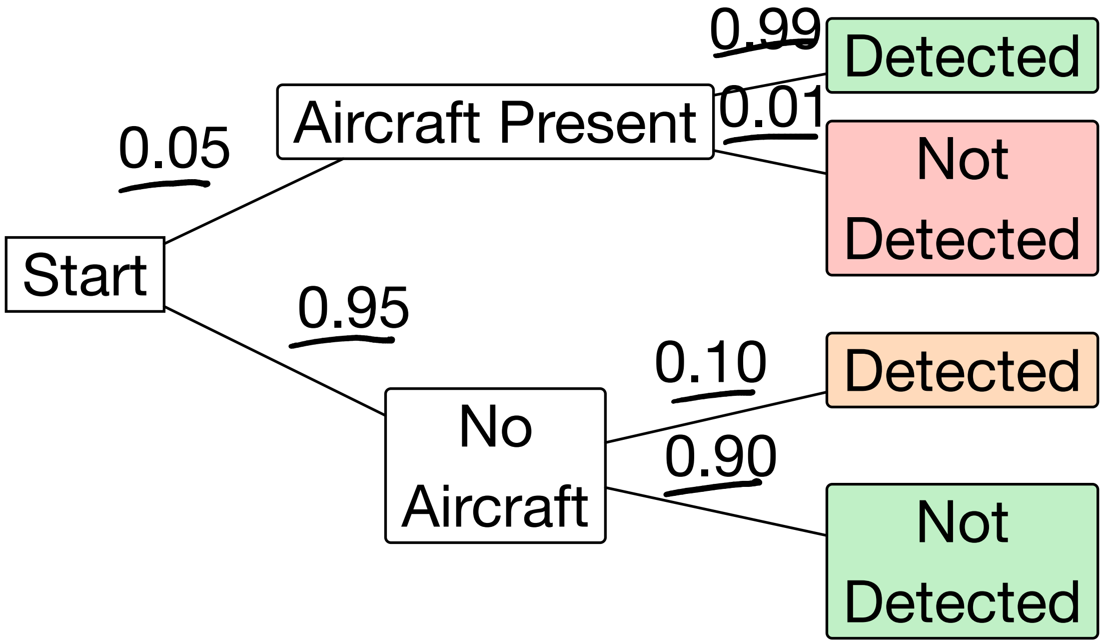
Questions:

1. What is the probability of a **false alarm**?
2. What is the probability of **missed detection**?

Bertsekas and Tsitsiklis

2.2 Visualizing with a Tree Diagram



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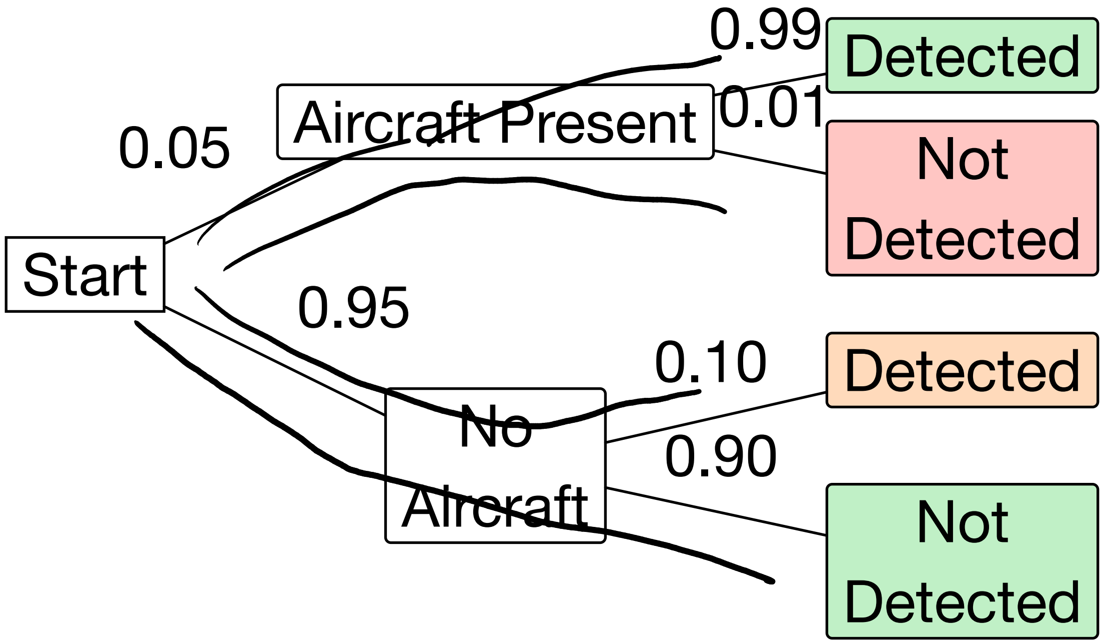
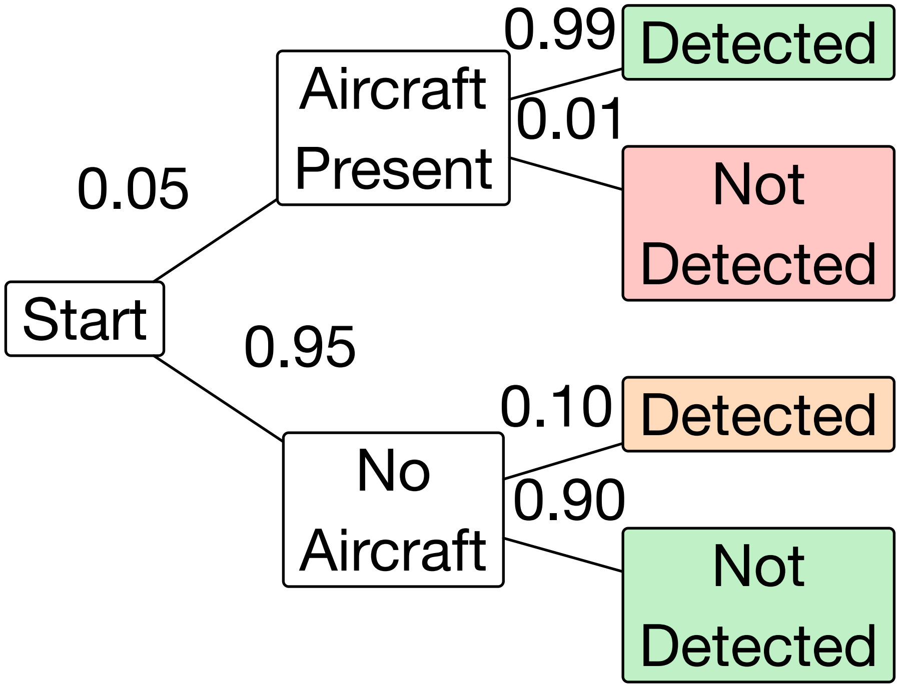


Each **path** from start to end represents a possible outcome.

2.3 Defining Events



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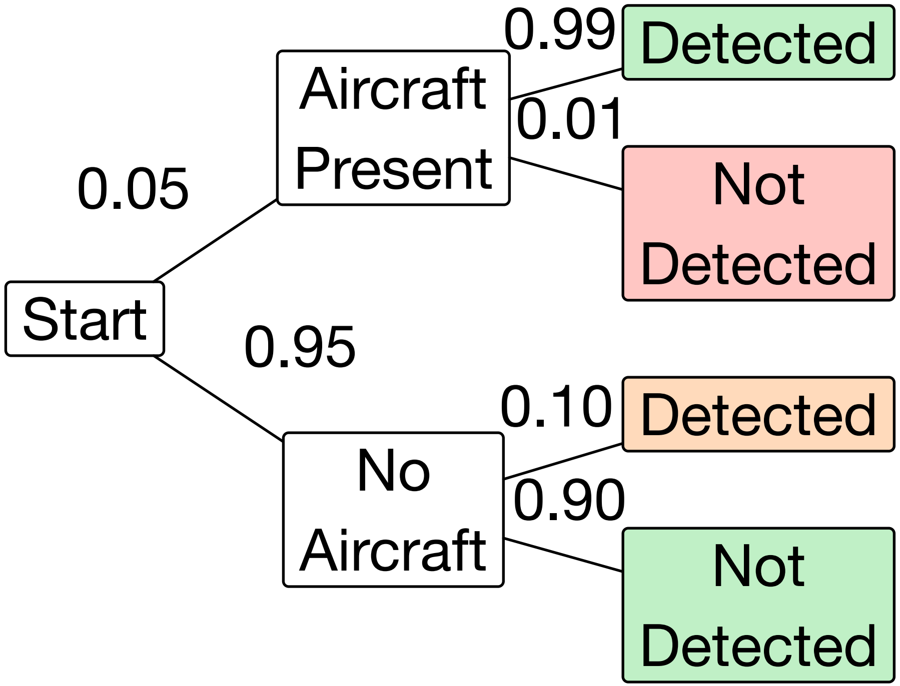

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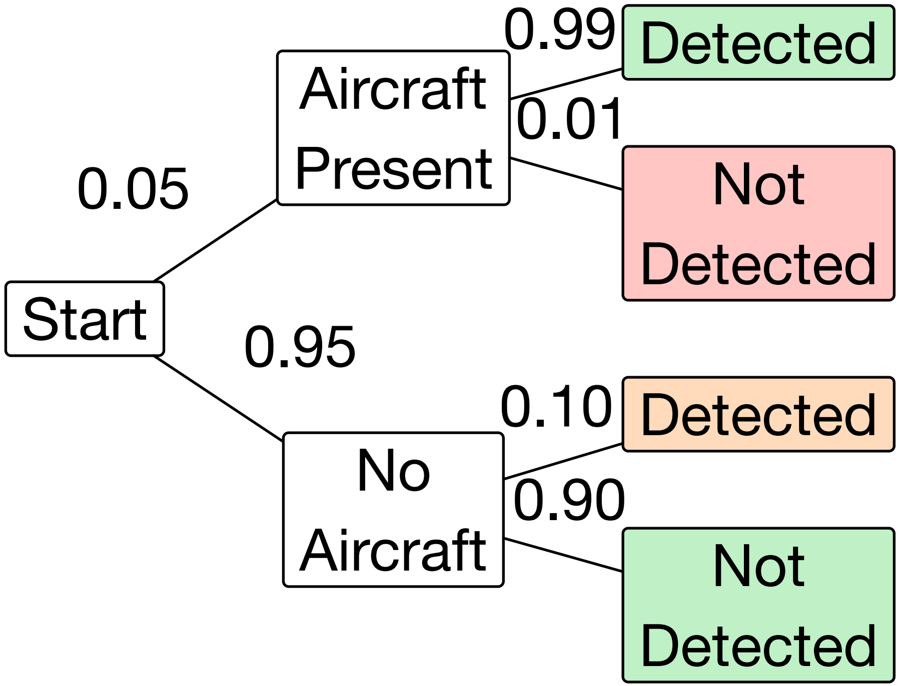

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False alarm: $A^c \cap B$

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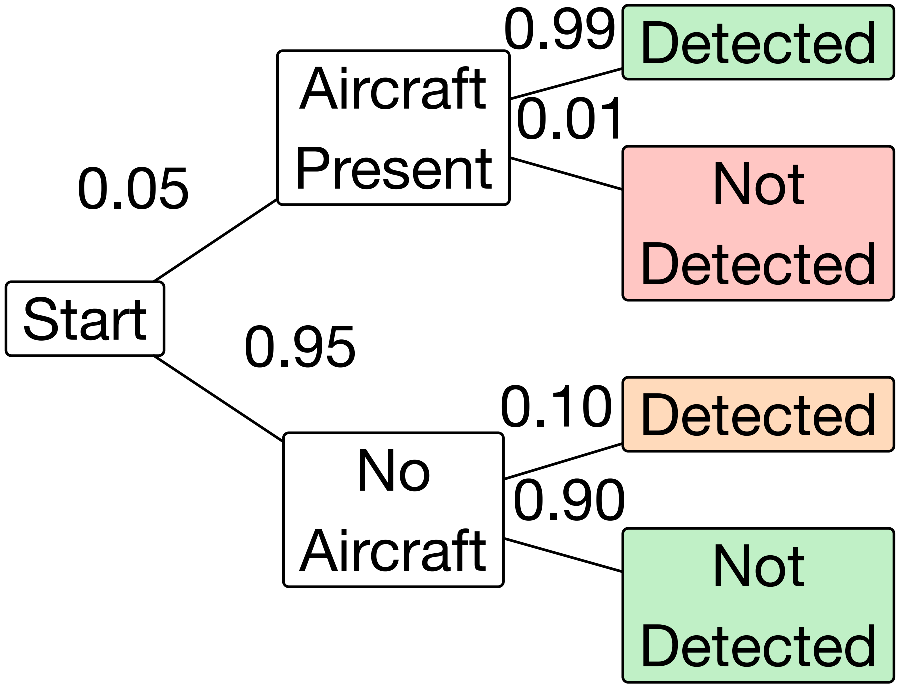

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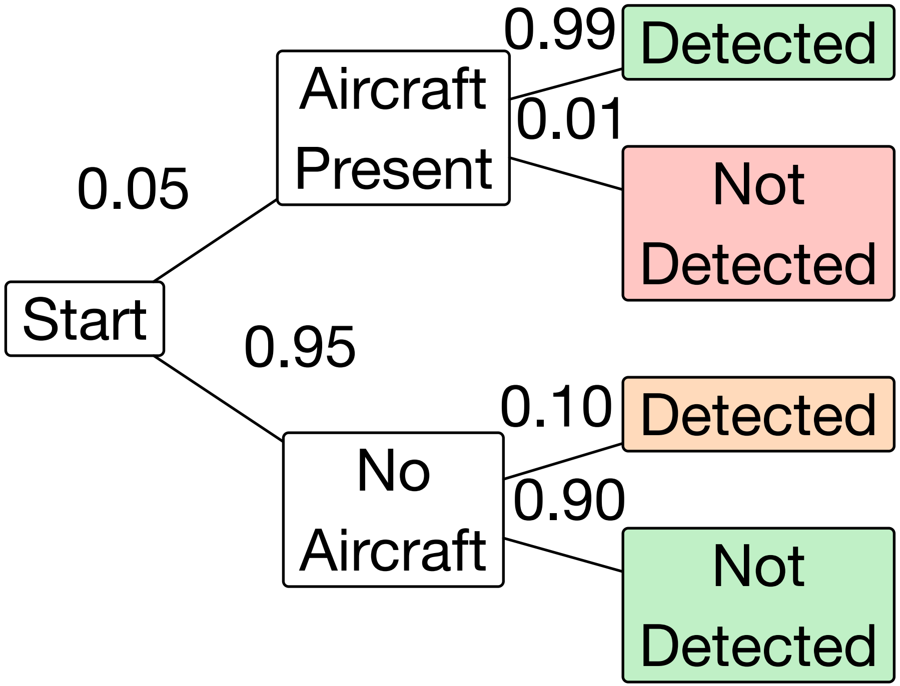
False alarm: $A^c \cap B$

Missed detection: $A \cap B^c$

2.4 Calculating False Alarm



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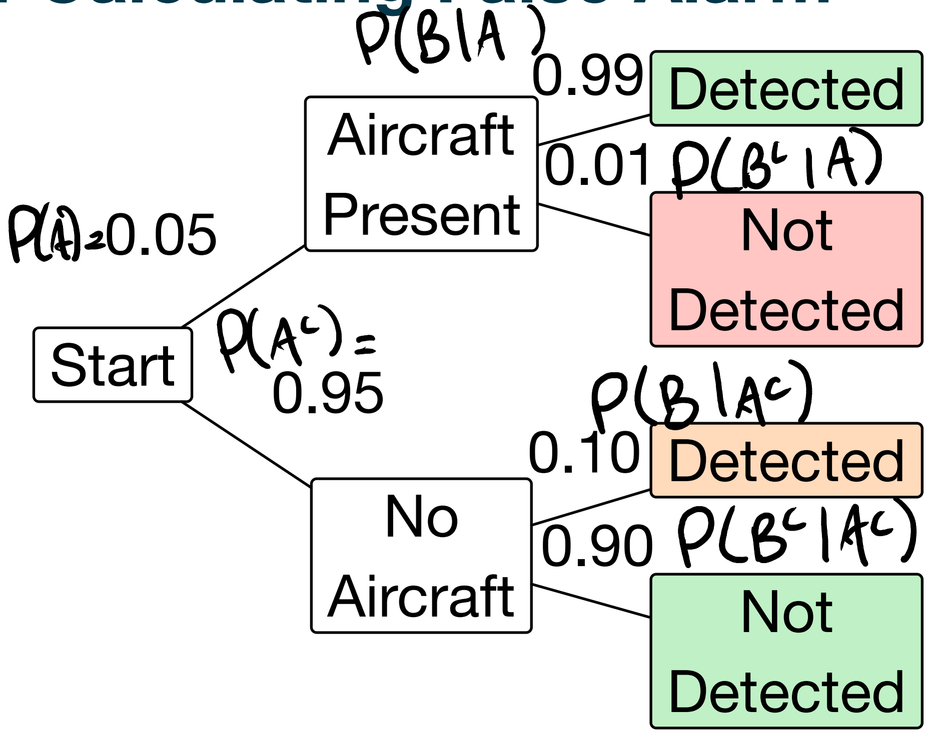

Use the multiplication rule:

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\begin{aligned}
P(\text{false alarm}) &= P(A^c \cap B) \\
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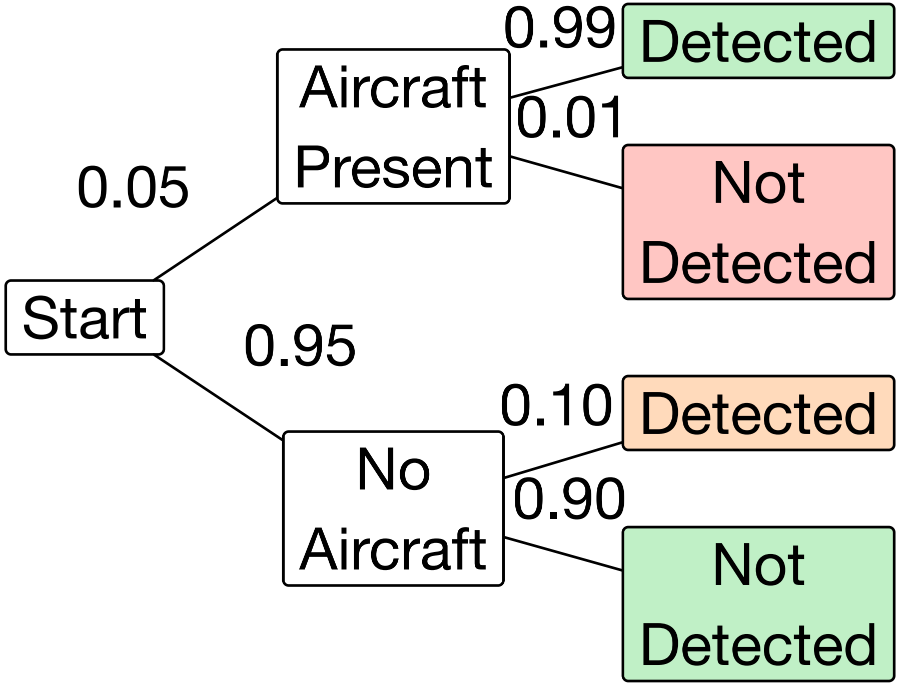

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$$= 0.95 \cdot 0.10$$

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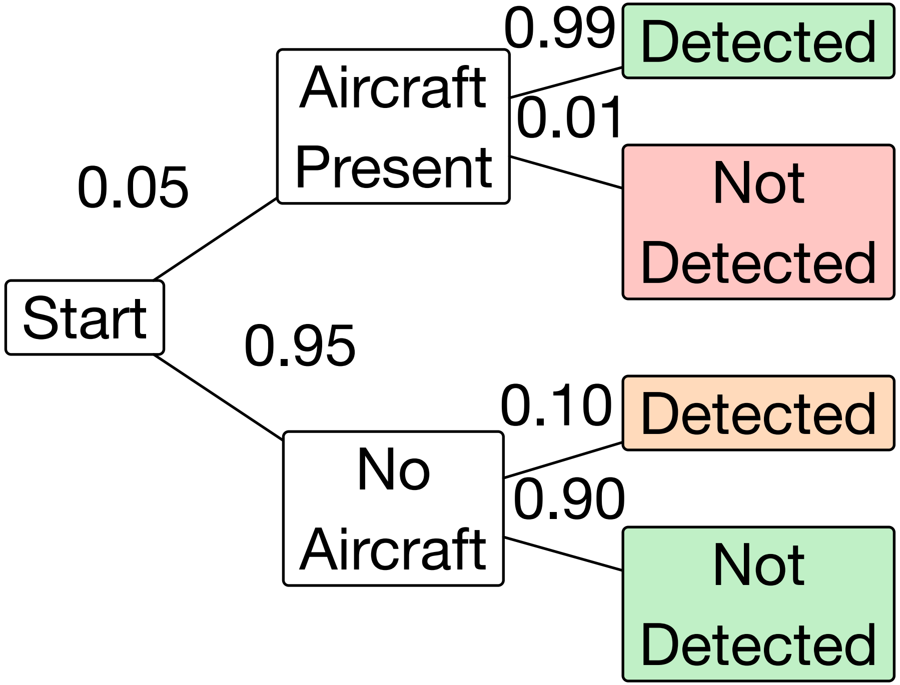

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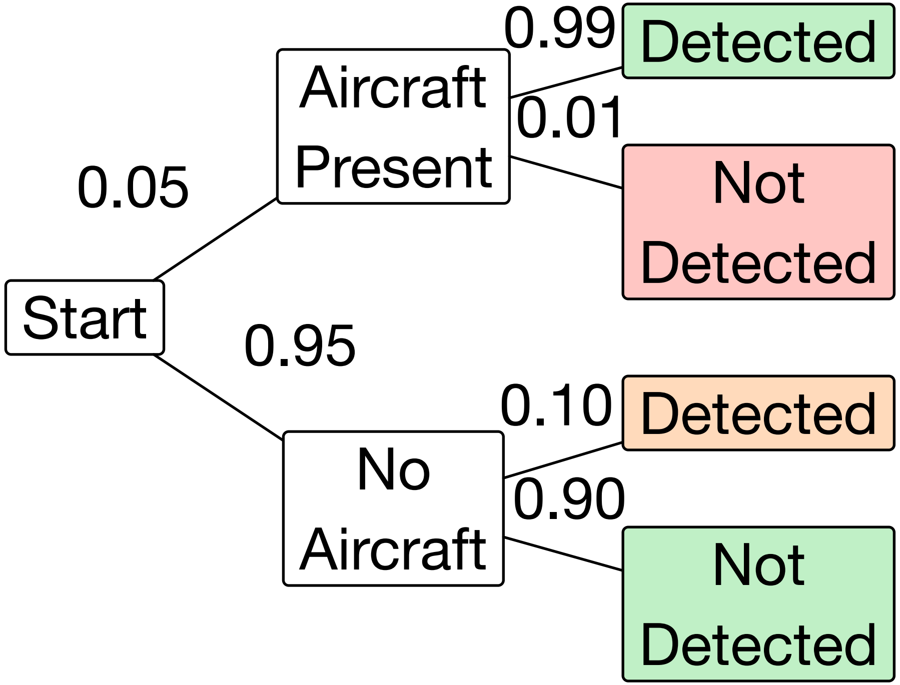

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About **9.5% chance** of a false alarm.

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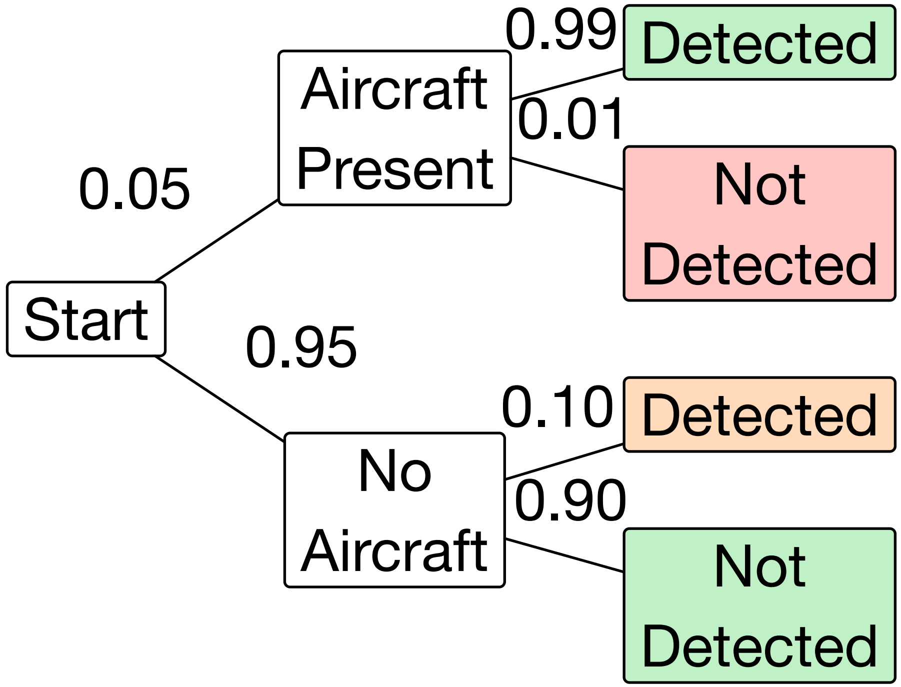

Similarly:

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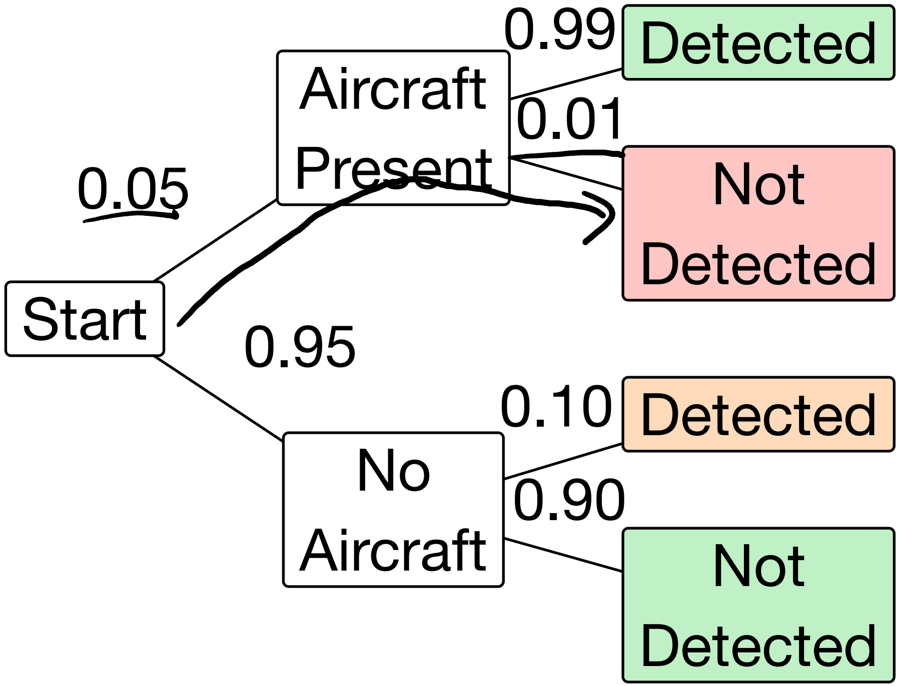

Start

0.05

Aircraft Present

0.99 Detected

0.01 Not Detected

0.95

No Aircraft

0.10 Detected

0.90 Not Detected

Similarly:

$$
\begin{aligned}
P(\text{missed detection}) &= P(\underline{A} \cap \underline{B^c}) \\
&= P(A) \cdot P(B^c \mid A)
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$$

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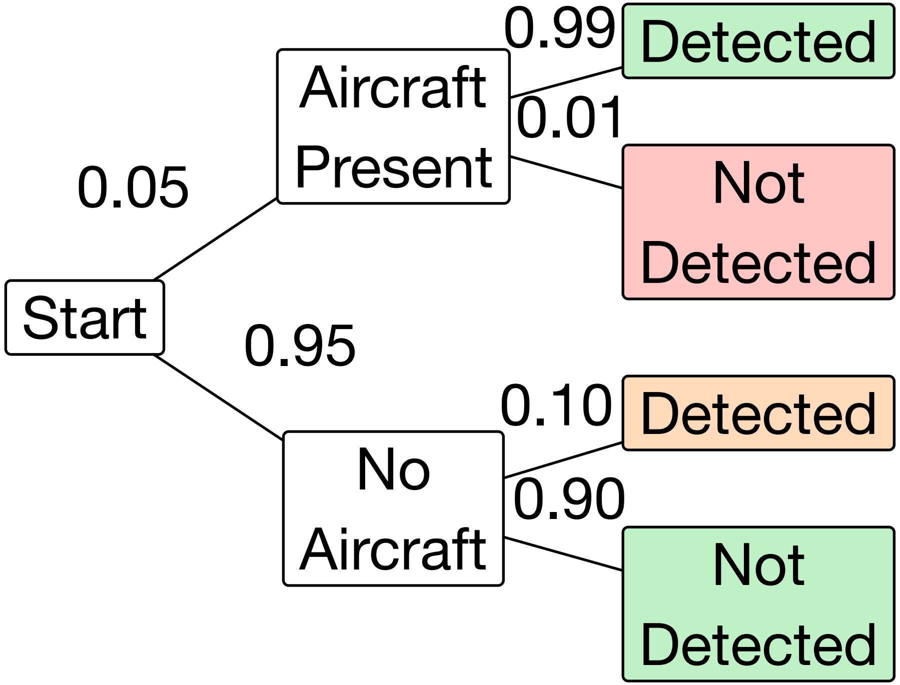

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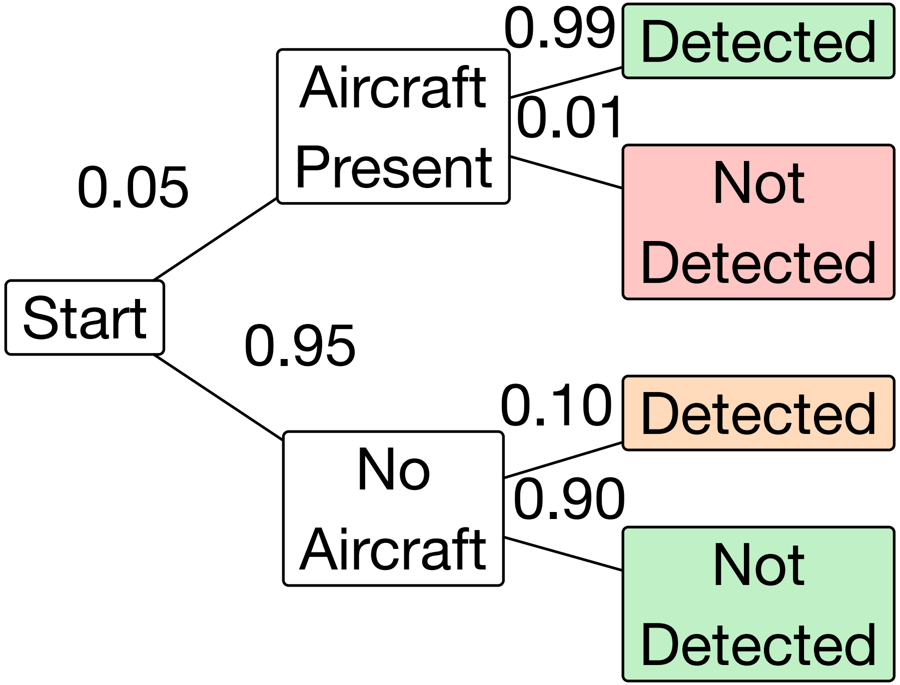

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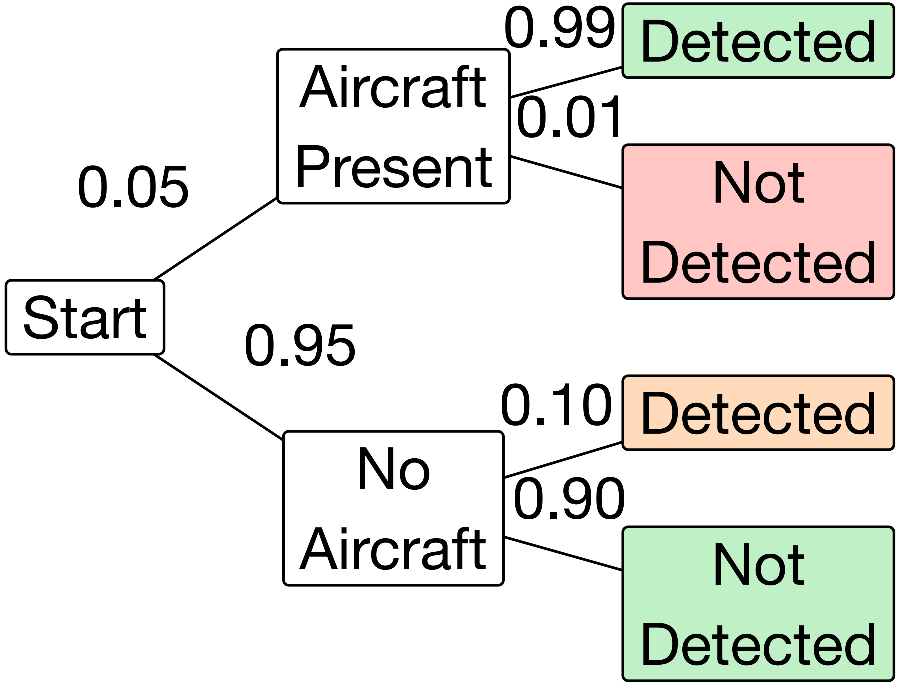

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$$= 0.05 \cdot 0.01$$

$$= 0.0005$$

Only **0.05% chance** of missing a present aircraft.

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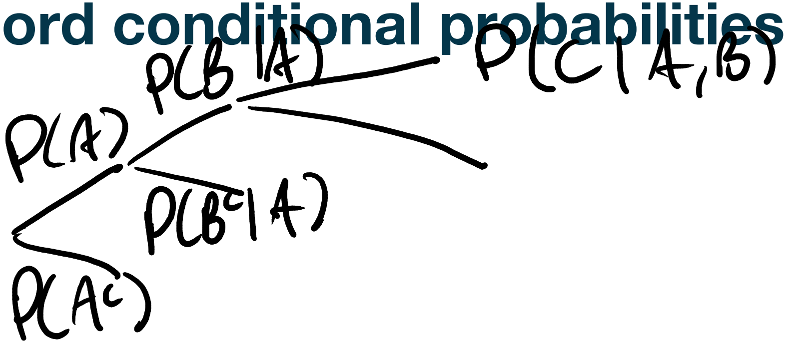

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This is exactly the multiplication rule, visualized!

3. Problem-Solving Strategies

3.1 Strategy: “How Can It Happen?”

When solving probability problems, ask: **“How can this event happen?”**

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Event: “End at square 4 with bonus roll, starting from 0”

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First guess: $R + B = 4$ and $B > 0$

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Equivalent simpler event: $R = 3$ and $B = 1$

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Example: Board Game

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First guess: $R + B = 4$ and $B > 0$

Equivalent simpler event: $R = 3$ and $B = 1$

Check both directions:

- If $\underline{R = 3}, \underline{B = 1}$, then $\underline{R + B = 4}$ and $\underline{B > 0}$ ✓
- If $\underline{R + B = 4}, \underline{B > 0}$, then $\underline{R = 3, B = 1}$ ✓

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- Condition on different events
- Use complement vs. direct calculation
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- Use complement vs. direct calculation
- Break into cases in different ways

If one approach gets messy, try another!

3.3 Strategy: Multiple Approaches

There's rarely one "right" way to solve a problem.

Try different decompositions:

- Condition on different events
- Use complement vs. direct calculation
- Break into cases in different ways

If one approach gets messy, try another!

Note: Like programming: creative process, not formula-based

Matloff

3.4 Chain Rule (Extended Multiplication)

For multiple events:

$$P(A_1 \cap A_2 \cap A_3) = \underline{P(A_1)} \cdot P(A_2 \mid \overset{\downarrow}{A_1}) \cdot P(A_3 \mid \overset{\downarrow}{A_1} \cap A_2)$$

$$= P(A_1) \cdot P(A_3 \mid \underset{\uparrow}{A_1}) \cdot P(A_2 \mid \underset{\uparrow}{A_1} \cap A_3)$$

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In general:

$$P\left(\bigcap_{i=1}^{\boxed{n}} A_i\right) = P(A_1) \cdot P(\underline{A_2} \mid \underline{A_1}) \cdot P(\underline{A_3} \mid \underline{A_1 A_2}) \cdot \ldots \cdot P(\underline{A_n} \mid \underline{A_1 \ldots A_{n-1}})$$

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This “chains” conditional probabilities together.

3.5 Example: Three Cards

Draw 3 cards without replacement. $P(\text{all three are hearts})$?

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$$
\begin{aligned}
P(\underline{H_1} \cap \underline{H_2} \cap \underline{H_3}) &= P(\underline{H_1}) \cdot P(H_2 \mid \underline{H_1}) \cdot P(\underline{H_3} \mid \underline{H_1 H_2}) \\
&= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \\
&= \frac{1716}{132600} \approx 0.0129
\end{aligned}
$$

3.6 Conditional Probability is a Probability

$P(\cdot \mid B)$ satisfies all three axioms:

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   $$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

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   ✓

So all our probability rules apply to conditional probabilities too!

3.7 Common Mistakes with Conditional Probability

Two hugely common errors:

1. $P(A \mid B) \neq P(B \mid A)$ (confusion of the inverse)
2. $P(A \underset{\substack{\uparrow \\ \text{joint}}}{\text{and}} \underset{?.}{\overline{B)}} \neq P(A \underset{\substack{\uparrow \\ \text{conditional}}}{\mid} B)$ ✓ (confusing "joint" and conditional)

3.8 Mistake 1: Confusion of the Inverse

$P(A \mid B) \neq P(B \mid A)$ in general!

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Example:

- $P(\text{has fever} \mid \text{has flu}) \approx \underline{0.9}$ (most flu patients have fever)
- $P(\text{has flu} \mid \text{has fever}) \approx \underline{0.1}$ (most fevers aren’t from flu)

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Example:

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- $P(\text{has flu} \mid \text{has fever}) \approx 0.1$ (most fevers aren’t from flu)

These condition on **different events**, so they’re completely different probabilities.

3.9 Mistake 2: Joint vs. Conditional

$P(A \text{ and } B) \neq P(A \mid B)$ in general!

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Roll two dice. Let X = first die, S = sum. Consider $X = 1$ and $S = 6$:

- $P(X = 1 \text{ and } S = 6) = \frac{1}{36}$
  - Fraction of **all** rolls where both happen

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  - Fraction of **all** rolls where both happen
- $P(X = 1 \mid \underline{S = 6}) = \boxed{\frac{1}{5}}$
  - Fraction of **$S = 6$ rolls** where $X = 1$ also happens

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  - Fraction of **all** rolls where both happen
- $P(X = 1 \mid S = 6) = \frac{1}{5}$
  - Fraction of $S = 6$ **rolls** where $X = 1$ also happens

Note: Use the notebook view: joint probability counts all lines, conditional counts only non-NA lines

3.10 Connecting to Code

Python:

```python
import random

def simulate_conditional_prob(n_trials=100000
    count_first_ace = 0
    count_both_aces = 0

    for _ in range(n_trials):
        deck = list(range(52))  # 0-3 are Aces
        random.shuffle(deck)

        if deck[0] < 4:  # First card is an Ace
            count_first_ace += 1
            if deck[1] < 4:  # Second card is also an Ace
                count_both_aces += 1

    # P(second ace | first ace) = rate among trials where first is ace
    conditional_prob = count_both_aces / count_first_ace
    return conditional_prob

print(f"P(2nd Ace | 1st Ace) simulated: {simulate_conditional_prob():.4f}")
print(f"P(2nd Ace | 1st Ace) theoretical: {3/51:.4f}")
```

$$\frac{P(A \cap B)}{P(B)}$$

3.10 Connecting to Code

R:

```r
simulate_conditional_prob <- function(n_trials = 100000) {
  count_first_ace <- 0
  count_both_aces <- 0

  for (i in 1:n_trials) {
    deck <- sample(1:52)  # 1-4 are Aces

    if (deck[1] <= 4) {  # First card is an Ace
      count_first_ace <- count_first_ace + 1
      if (deck[2] <= 4) {  # Second card is also an Ace
        count_both_aces <- count_both_aces + 1
      }
    }
  }

  # P(second ace | first ace) = rate among trials where first is ace
  conditional_prob <- count_both_aces / count_first_ace
  return(conditional_prob)
}

cat("P(2nd Ace | 1st Ace) simulated:", simulate_conditional_prob(), "\n")
cat("P(2nd Ace | 1st Ace) theoretical:", 3/51, "\n")
```

3.11 Summary

Concept	**Formula**
Conditional Probability	$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
Multiplication Rule	$P(A \cap B) = P(A) \cdot P(B \mid A)$
Chain Rule	$P(A_1 \cap \ldots \cap A_n) = \prod P(A_i \mid A_1 \ldots A_{i-1})$

3.13 Recap

Today we covered:

- Conditional probability: probability given new information
- Multiplication rule: $P(A \cap B) = P(A) \cdot P(B \mid A)$
- Chain rule extends to multiple events
- $P(A \mid B) \neq P(B \mid A)$ - order matters!
- Next: Bayes' rule and total probability

$P(A \cap B) = P(A) \cdot P(B)$

$P(B) = P(B \mid A)$