1. Conditional Probability

1.1 Learning Objectives

By the end of this lecture, you will be able to:

- Define and compute conditional probabilities
- Apply the multiplication rule
- Understand how conditioning reduces the sample space
- Use conditional probability in sequential experiments

1.2 Motivating Example

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$$P(\{6\} \mid \text{even}) = \frac{1}{3}$$

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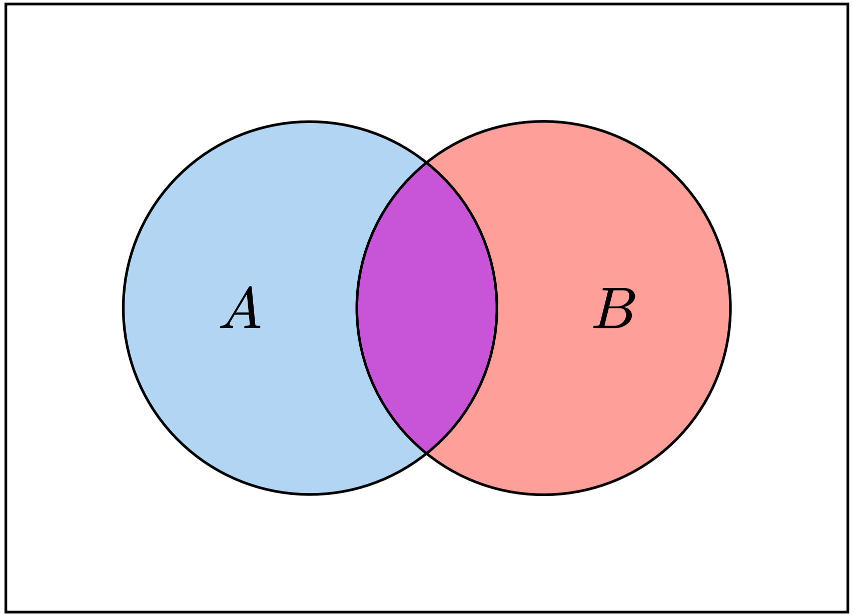
Definition: Conditional Probability

The conditional probability of A given B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.

1.4 Visualizing Conditional Probability



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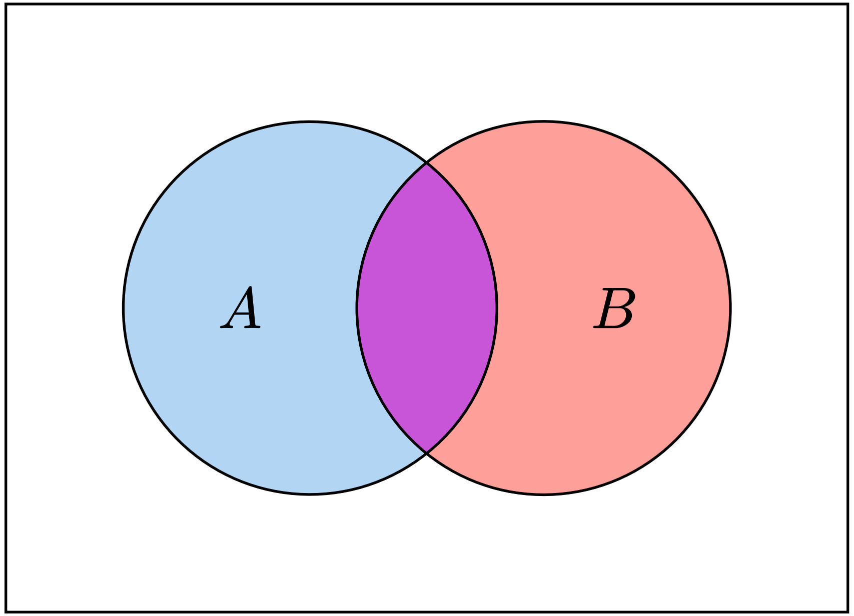

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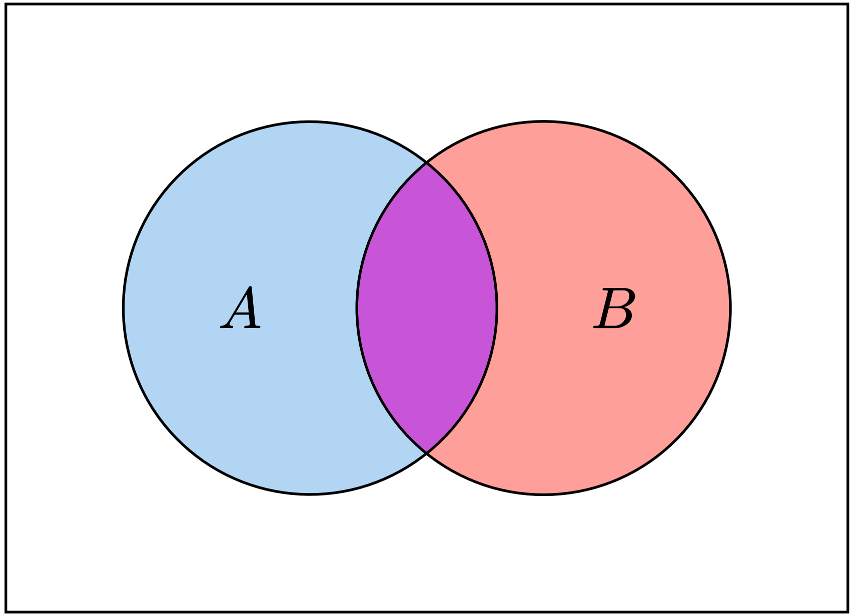

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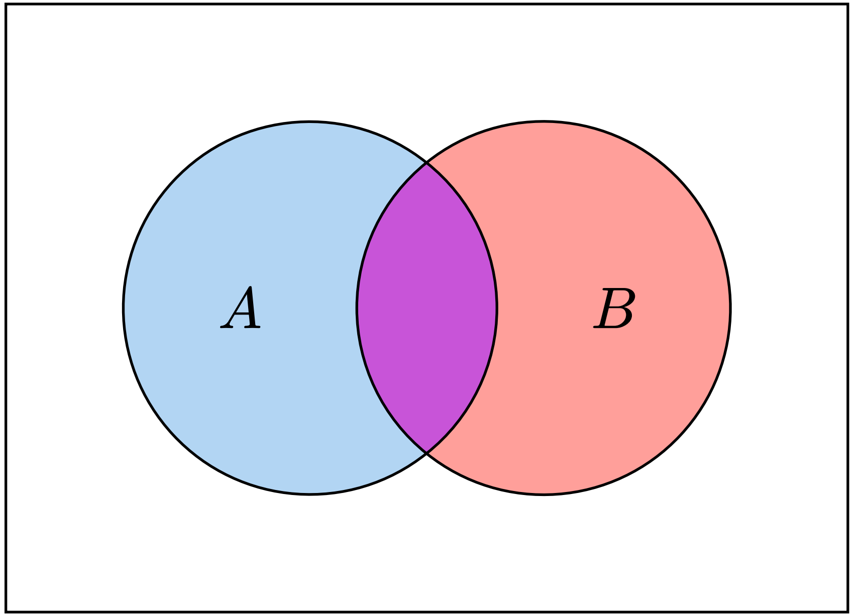

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Note: We only look at lines where B occurred, then ask: among those, what fraction also has A?

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Line	X	S	X = 1 and S = 6	X = 1 | S = 6
1	3	8	No	NA
2	1	6	Yes	Yes
3	2	7	No	NA
4	5	6	No	No
5	1	4	No	NA
6	4	6	No	No

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4	5	6	No	No
5	1	4	No	NA
6	4	6	No	No

- $P(X = 1 \text{ and } S = 6) = \frac{1}{36}$ (1 Yes out of 6 lines total)
- $P(X = 1 \mid S = 6) = \frac{1}{3}$ (1 Yes out of 3 non-NA lines)

1.7 Example: Drawing Cards

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There are 13 hearts, and exactly 1 is an Ace.

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Rearranging the definition:

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$

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Note: Very useful for sequential experiments!

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$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2 \mid A_1)$$
$$= \frac{4}{52} \cdot \frac{3}{51}$$
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After drawing an Ace, only 3 Aces remain among 51 cards.

2. Tree-Based Modeling

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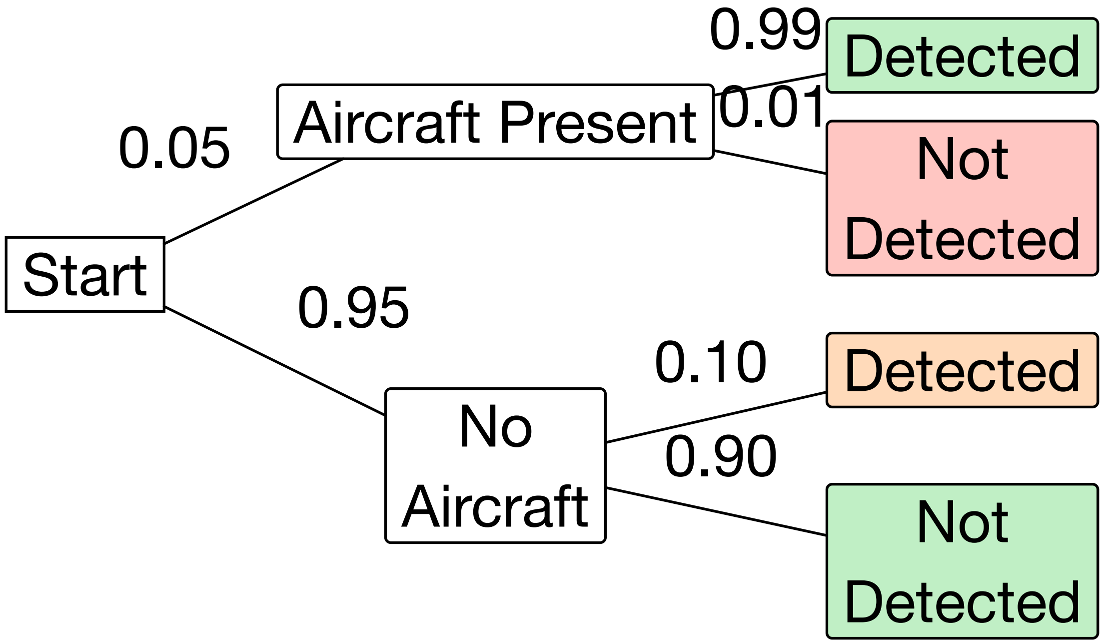
Questions:

1. What is the probability of a **false alarm**?
2. What is the probability of **missed detection**?

Bertsekas and Tsitsiklis

2.2 Visualizing with a Tree Diagram



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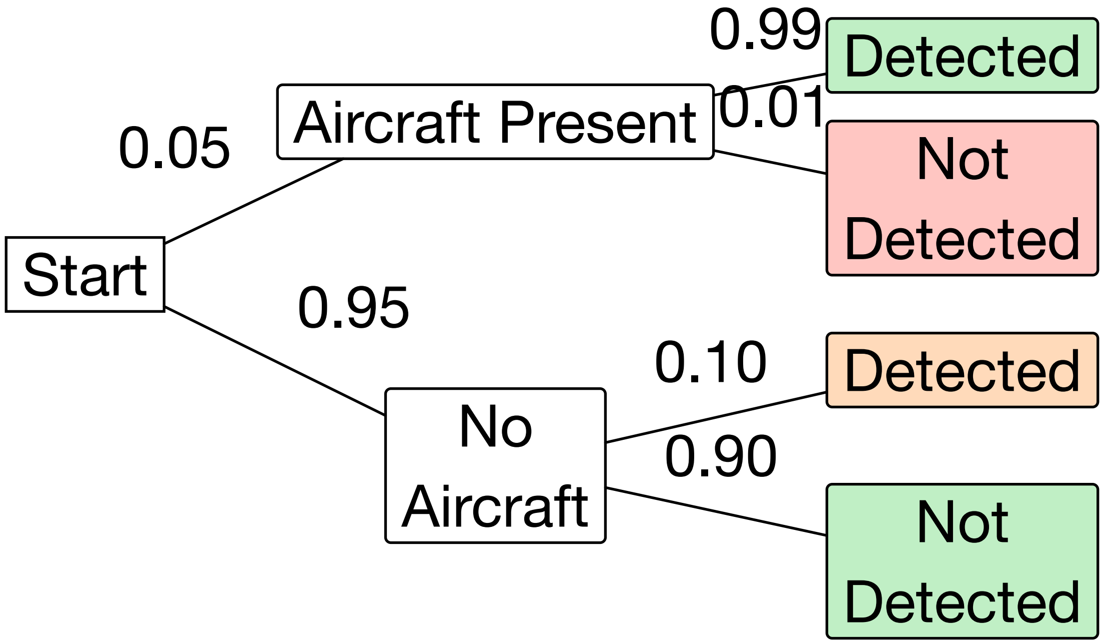
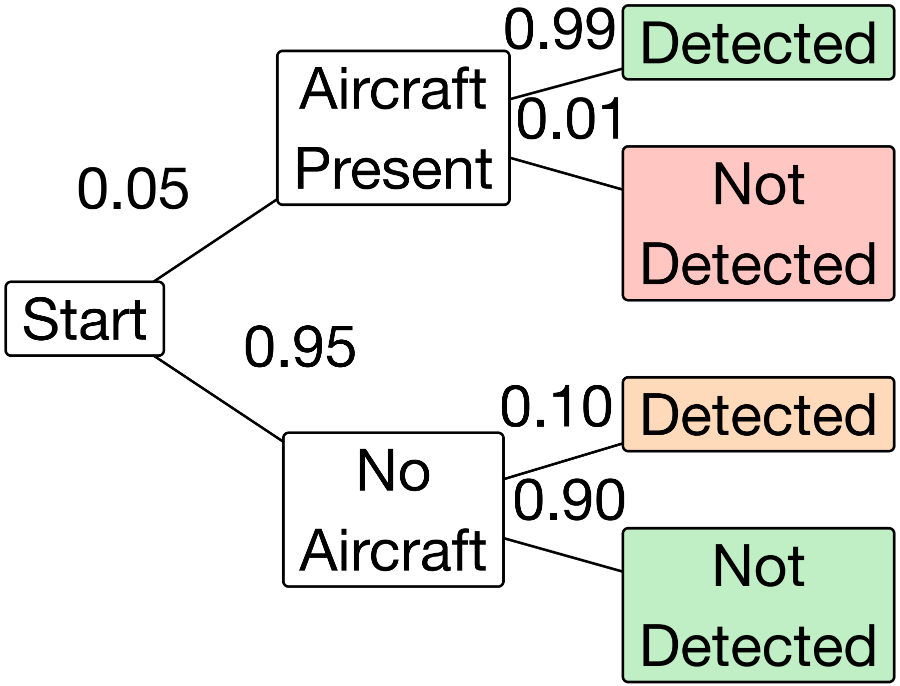


Each **path** from start to end represents a possible outcome.

2.3 Defining Events



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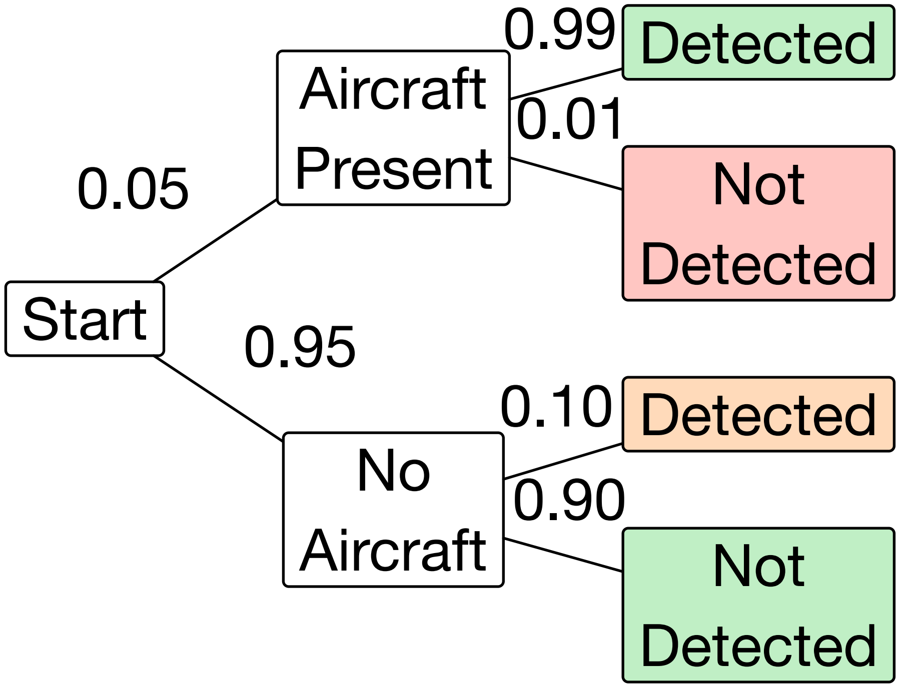

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Also consider the complements:

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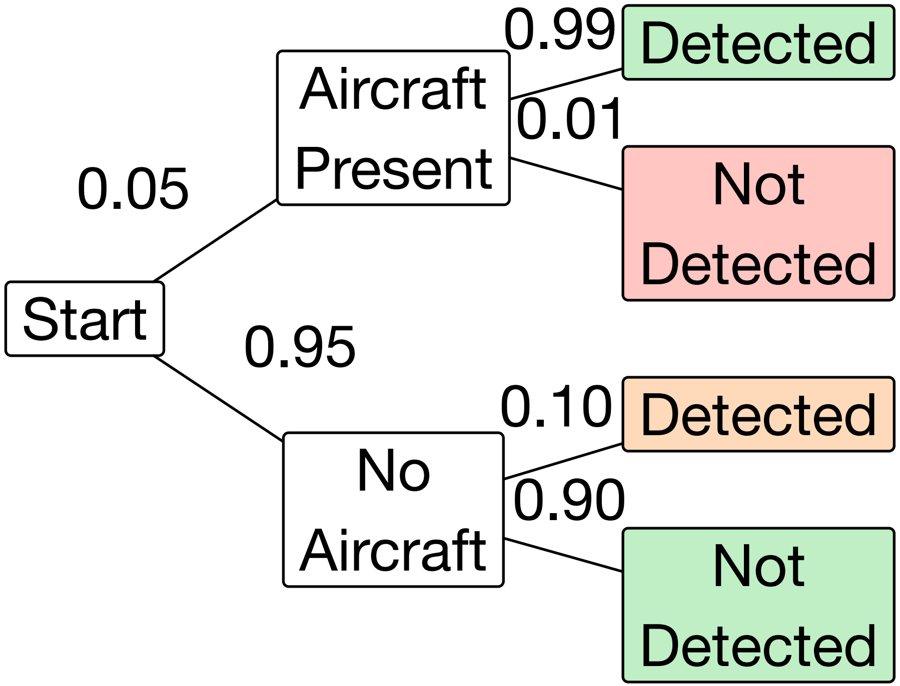

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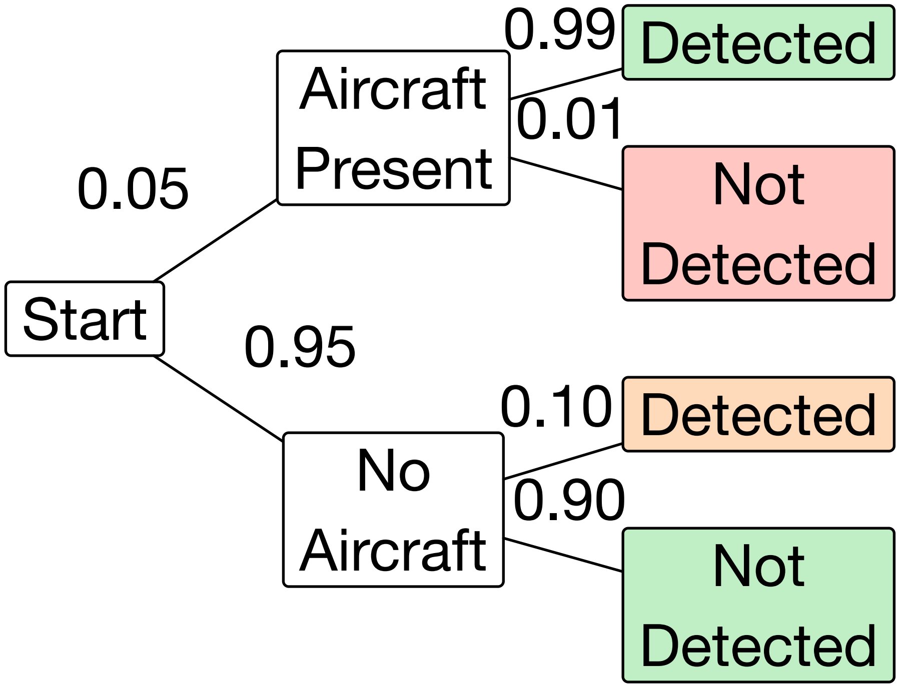

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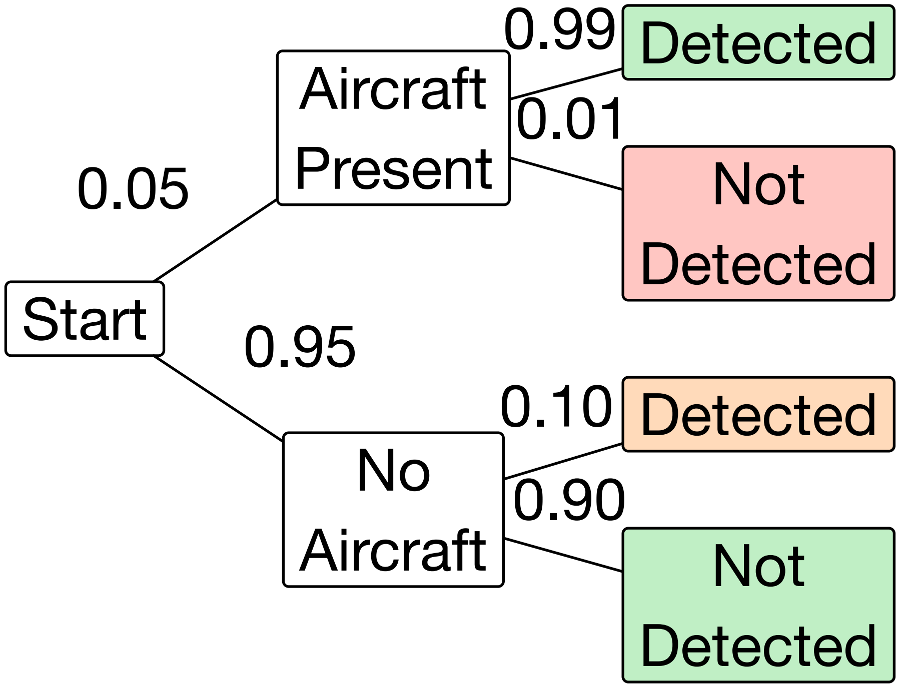
Missed detection: $A \cap B^c$

2.4 Calculating False Alarm



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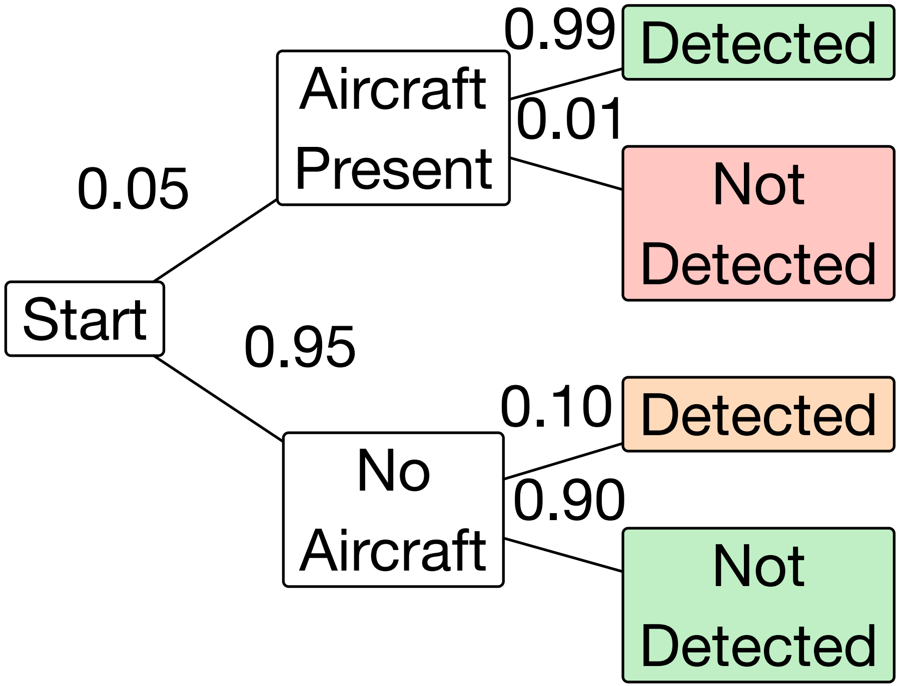

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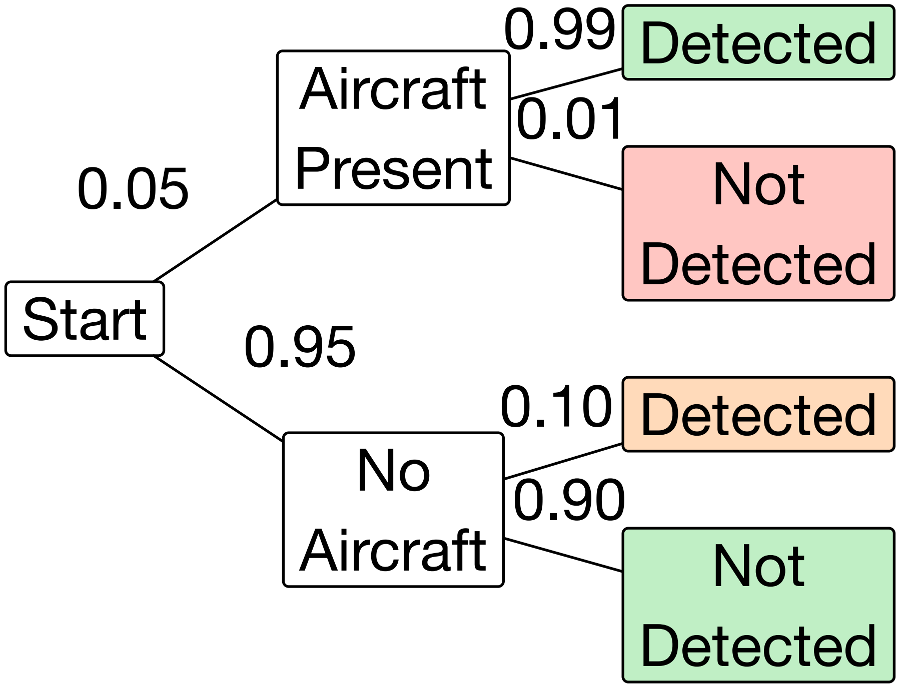

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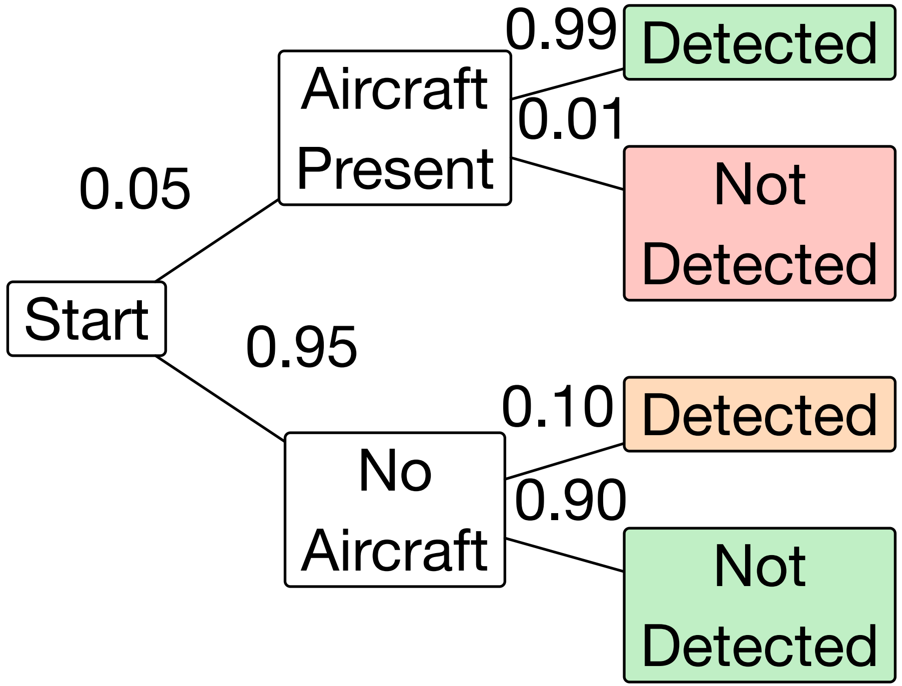

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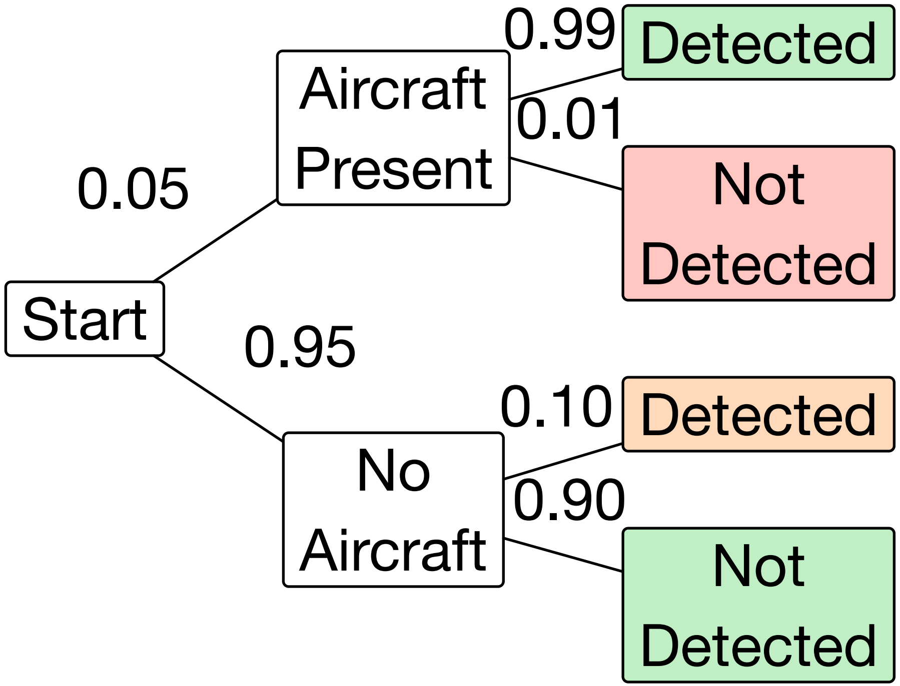

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About **9.5% chance** of a false alarm.

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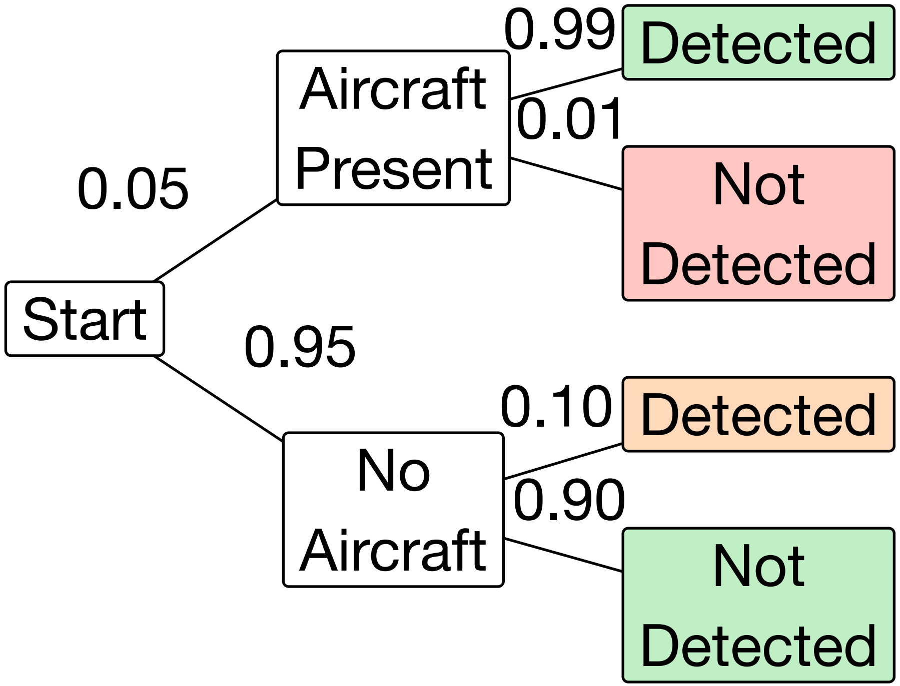

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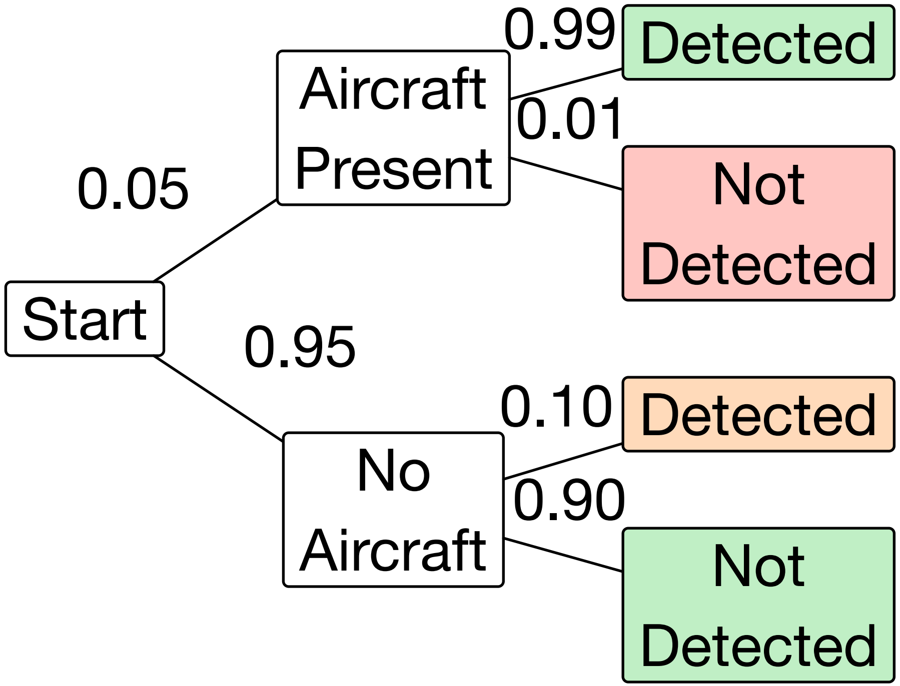

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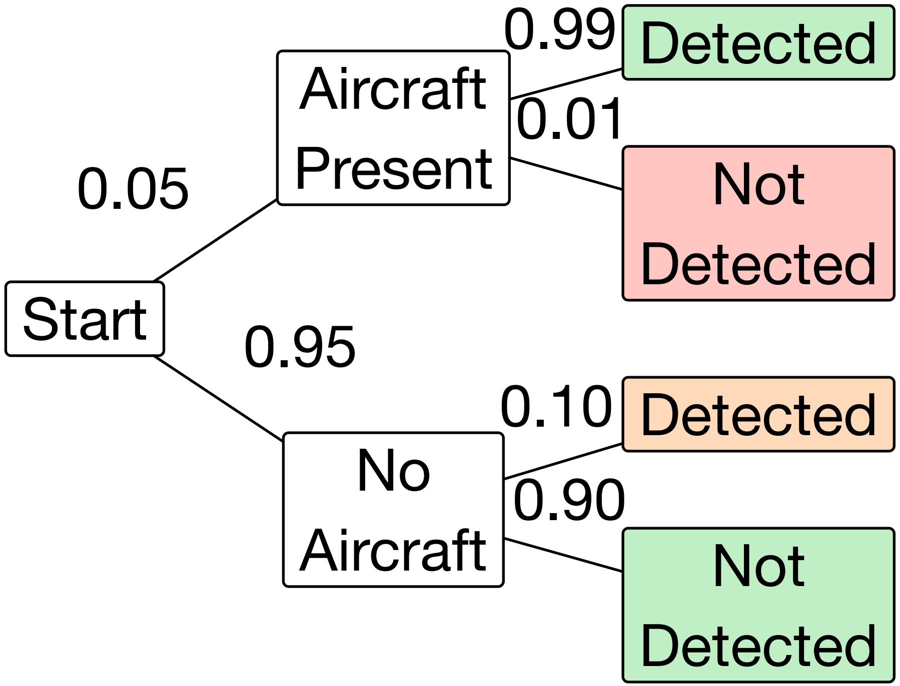

Similarly:

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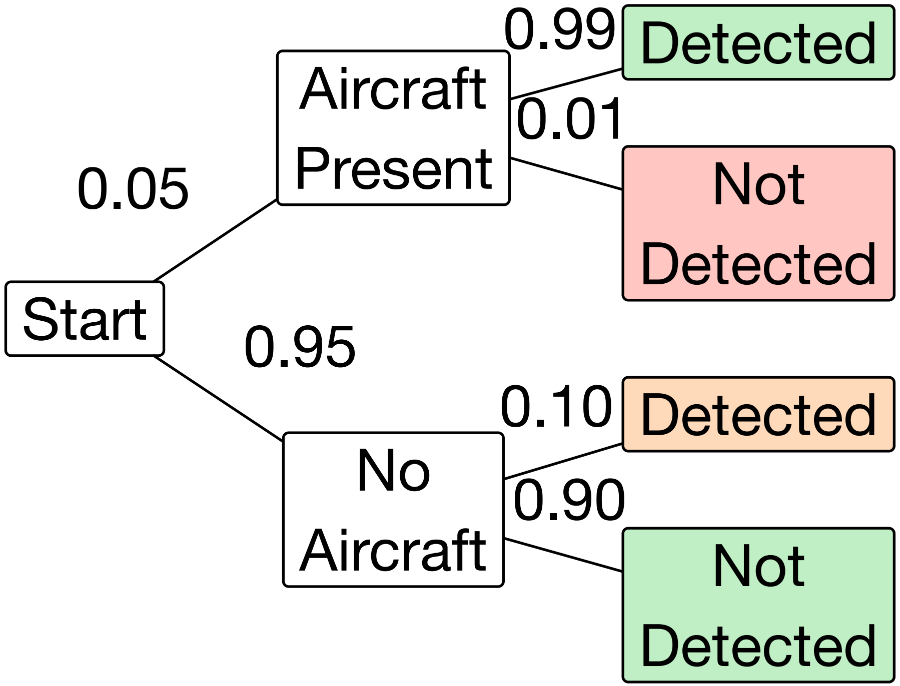

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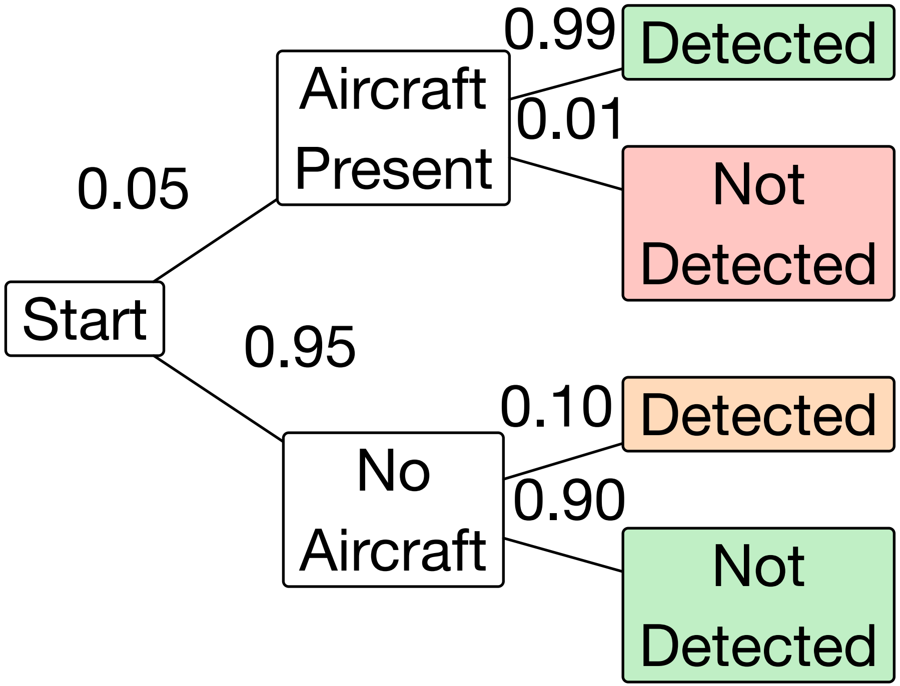

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Only **0.05% chance** of missing a present aircraft.

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This is exactly the multiplication rule, visualized!

3. Problem-Solving Strategies

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Equivalent simpler event: $R = 3$ and $B = 1$

Check both directions:

- If $R = 3, B = 1$, then $R + B = 4$ and $B > 0$ ✓
- If $R + B = 4, B > 0$, then $R = 3, B = 1$ ✓

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Note: Like programming: creative process, not formula-based

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3.4 Chain Rule (Extended Multiplication)

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In general:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 A_2) \cdot \ldots \cdot P(A_n \mid A_1 \ldots A_{n-1})$$

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This “chains” conditional probabilities together.

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\begin{aligned}
P(H_1 \cap H_2 \cap H_3) &= P(H_1) \cdot P(H_2 \mid H_1) \cdot P(H_3 \mid H_1 H_2) \\
&= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \\
&= \frac{1716}{132600} \approx 0.0129
\end{aligned}
$$

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1. **Non-negativity**: $P(A \mid B) \geq 0$ ✓
2. **Normalization**: $P(B \mid B) = \frac{P(B \cap B)}{P(B)} = 1$ ✓

3.6 Conditional Probability is a Probability

$P(\cdot \mid B)$ satisfies all three axioms:

1. **Non-negativity**: $P(A \mid B) \geq 0$ ✓
2. **Normalization**: $P(B \mid B) = \frac{P(B \cap B)}{P(B)} = 1$ ✓
3. **Additivity**: If A_1, A_2 disjoint:

$$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

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   $$P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$$

   ✓

So all our probability rules apply to conditional probabilities too!

3.7 Common Mistakes with Conditional Probability

Two hugely common errors:

1. $P(A \mid B) \neq P(B \mid A)$ (confusion of the inverse)
2. $P(A \text{ and } B) \neq P(A \mid B)$ (confusing joint and conditional)

3.8 Mistake 1: Confusion of the Inverse

$P(A \mid B) \neq P(B \mid A)$ in general!

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Example:

- $P(\text{has fever} \mid \text{has flu}) \approx 0.9$ (most flu patients have fever)
- $P(\text{has flu} \mid \text{has fever}) \approx 0.1$ (most fevers aren't from flu)

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These condition on **different events**, so they're completely different probabilities.

3.9 Mistake 2: Joint vs. Conditional

$P(A \text{ and } B) \neq P(A \mid B)$ in general!

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Roll two dice. Let X = first die, S = sum. Consider $X = 1$ and $S = 6$:

- $P(X = 1 \text{ and } S = 6) = \frac{1}{36}$
  - Fraction of **all** rolls where both happen

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  - Fraction of **all** rolls where both happen
- $P(X = 1 \mid S = 6) = \frac{1}{5}$
  - Fraction of **$S = 6$ rolls** where $X = 1$ also happens

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- $P(X = 1 \mid S = 6) = \frac{1}{5}$
  - Fraction of **$S = 6$ rolls** where $X = 1$ also happens

Note: Use the notebook view: joint probability counts all lines, conditional counts only non-NA lines

3.10 Connecting to Code

Python:

Python

```python
import random

def simulate_conditional_prob(n_trials=100000
    count_first_ace = 0
    count_both_aces = 0

    for _ in range(n_trials):
        deck = list(range(52))  # 0-3 are Aces
        random.shuffle(deck)

        if deck[0] < 4:  # First card is an Ace
            count_first_ace += 1
            if deck[1] < 4:  # Second card is also an Ace
                count_both_aces += 1

    # P(second ace | first ace) = rate among trials where first is ace
    conditional_prob = count_both_aces / count_first_ace
    return conditional_prob

print(f"P(2nd Ace | 1st Ace) simulated: {simulate_conditional_prob():.4f}")
print(f"P(2nd Ace | 1st Ace) theoretical: {3/51:.4f}")
```

3.10 Connecting to Code

R:

```
simulate_conditional_prob <- function(n_trials = 100000) {
  count_first_ace <- 0
  count_both_aces <- 0

  for (i in 1:n_trials) {
    deck <- sample(1:52)  # 1-4 are Aces

    if (deck[1] <= 4) {  # First card is an Ace
      count_first_ace <- count_first_ace + 1
      if (deck[2] <= 4) {  # Second card is also an Ace
        count_both_aces <- count_both_aces + 1
      }
    }
  }

  # P(second ace | first ace) = rate among trials where first is ace
  conditional_prob <- count_both_aces / count_first_ace
  return(conditional_prob)
}

cat("P(2nd Ace | 1st Ace) simulated:", simulate_conditional_prob(), "\n")
cat("P(2nd Ace | 1st Ace) theoretical:", 3/51, "\n")
```

3.11 Summary

Concept	Formula
Conditional Probability	$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
Multiplication Rule	$P(A \cap B) = P(A) \cdot P(B \mid A)$
Chain Rule	$P(A_1 \cap \ldots \cap A_n) = \prod P(A_i \mid A_1 \ldots A_{i-1})$

3.13 Recap

Today we covered:

- Conditional probability: probability given new information
- Multiplication rule: $P(A \cap B) = P(A) \cdot P(B \mid A)$
- Chain rule extends to multiple events
- $P(A \mid B) \neq P(B \mid A)$ - order matters!
- Next: Bayes' rule and total probability