0.1 Learning Objectives

By the end of this lecture, you will be able to:

- Apply properties of expectation: scaling, shift, additivity, LOTUS
- Recognize and work with Uniform, Bernoulli, and Geometric distributions
- Use indicator random variables to solve counting problems
- Apply linearity of expectation to simplify complex calculations

1. Properties of Expectation

1.1 Quick Recall

Recall: For a discrete random variable X with PMF $p_{X(x)}$:

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Definition: Expectation of a Constant

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Note: This seems obvious, but it's useful in proofs!

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Example: Doubling a Die

If X is a fair die roll, $E(X) = \frac{7}{2}$.

If you double the result: $E(2X) = 2 \cdot \frac{7}{2} = 7$

1.4 Property 3: Shift

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Combining scaling and shift:

$$E(aX+b) = aE(X) + b$$

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Combining with scaling gives us **linearity**:

$$E(aX + bY) = aE(X) + bE(Y)$$

2. Introducing Distributions

2.1 From Data to Histograms

Suppose we survey 50 students: “How many classes are you taking?”

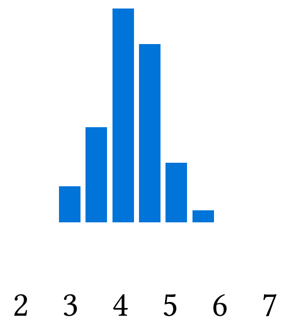
Raw data:

Student	Classes
1	4
2	5
3	4
4	3
5	4
⋮	⋮
50	5

Frequency table:

Classes	Count
2	3
3	8
4	18
5	15
6	5
7	1

Histogram:



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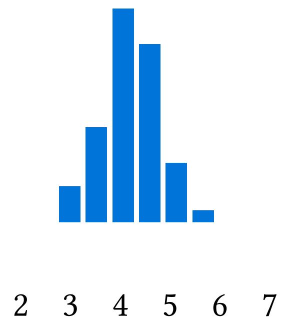
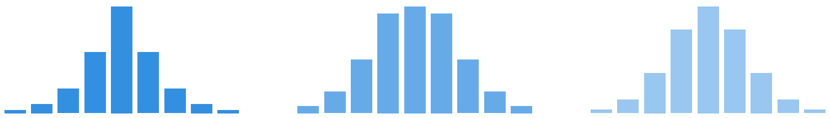


The histogram shows the **shape** of the data.

2.2 Patterns in Data: Family Resemblance

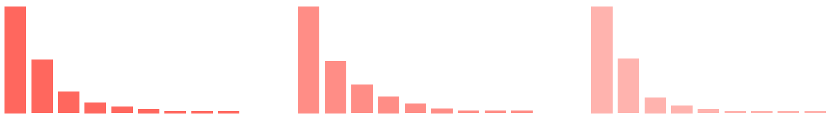
Different datasets, but similar shapes:

"Bell-shaped"



- Symmetric around center
- Most data near the middle
- Tails drop off smoothly

"Right-skewed"



- Peak on the left
- Long tail to the right
- Common: income, wait times

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A mathematical description of a random variable's possible values and their probabilities.

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Definition: Probability Distribution

A mathematical description of a random variable's possible values and their probabilities.

Instead of working with raw data, we can work with the **idealized pattern**.

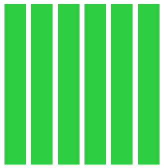
3. The Discrete Uniform Distribution

3.1 Equal Probability Outcomes

Definition: Discrete Uniform Distribution

$X \sim \text{Uniform}\{a, a+1, ..., b\}$ if X takes each value with equal probability:

$$P(X = k) = \frac{1}{b - a + 1}, \quad k = a, a+1, ..., b$$



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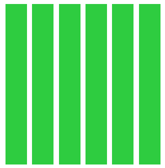

All bars equal height!

Mean:

$$E(X) = \frac{a+b}{2}$$

(midpoint of the range)

Variance:

$$\text{Var}(X) = \frac{(b-a+1)^2 - 1}{12}$$

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A fair die is $X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\}$

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Note: We computed this variance earlier using the definition — the formula is much faster!

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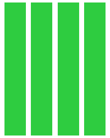
$\text{Uniform}\{1, 2, 3, 4\}$

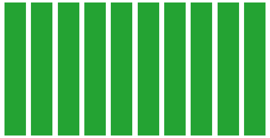
$a = 1$, $b = 4$



4 outcomes, each with prob $1/4$

$\text{Uniform}\{1, 2, \ldots, 10\}$

$a = 1$, $b = 10$



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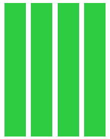

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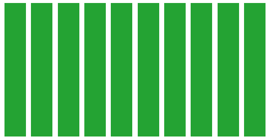

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Definition: Parameters

Values that specify a particular distribution within a family.

For Uniform: a (min) and b (max) are the **parameters**.

4. LOTUS: Law of the Unconscious Statistician

4.1 The Problem

Suppose $X \sim \text{Uniform}\{1, ..., 6\}$ (a die roll) and we want $E(X^2)$.

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X: 1, 2, 3, 4, 5, 6 Each with probability $\frac{1}{6}$

X^2: 1, 4, 9, 16, 25, 36

$$E(X^2) = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

4.2 LOTUS: The Easy Way

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$$E(g(X)) = \sum_{x} g(x) \cdot p_X(x)$$

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Key insight: Use the PMF of X, not the PMF of $g(X)$!

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$$E(X^2) = \sum_{x=1}^{6} x^2 \cdot \frac{1}{6} = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

Same answer, but we didn't need to find the PMF of X^2

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The "unconscious" approach gives the right answer.

4.4 LOTUS: Try It Yourself

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Talk to your neighbor and try to solve this problem.

Let $X \sim \text{Uniform}\{-2, -1, 0, 1, 2\}$.

Find $E(X^2)$.

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$$= \frac{1}{5}(4 + 1 + 0 + 1 + 4) = \frac{10}{5} = 2$$

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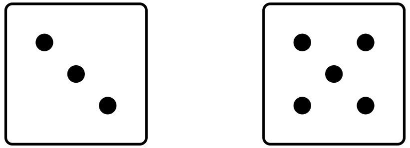
$$= \frac{1}{5}(4 + 1 + 0 + 1 + 4) = \frac{10}{5} = 2$$

Note: $E(X) = 0$ but $E(X^2) = 2$, so $E(X^2) \neq (E(X))^2$ in general!

5. Linearity in Action

5.1 Motivating Problem

You roll two fair dice and add up the faces.



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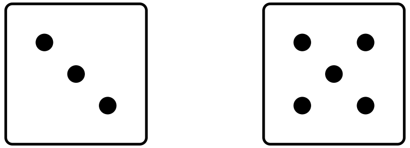

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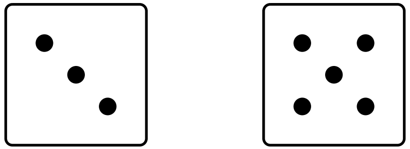

Question: What is the expected value of the sum?

Think for 30 seconds: How would you approach this?

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- $(1, 1) \to 2$
- $(1, 2) \to 3$
- $(1, 3) \to 4$
- $\vdots$
- $(6, 4) \to 10$
- $(6, 5) \to 11$
- $(6, 6) \to 12$

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Then compute: $E(X + Y) = \sum_{i=2}^{12} i \cdot P(X + Y = i)$

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Then compute: $E(X + Y) = \sum_{i=2}^{12} i \cdot P(X + Y = i)$

This is tedious! There must be a better way…

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Let X = first die, Y = second die.

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Step 1: Find $E(X)$ for a single die.

$$E(X) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

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$$E(X+Y) = E(X) + E(Y) = \frac{7}{2} + \frac{7}{2} = 7$$

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The expected sum of two dice is **7**

6. Indicator Random Variables and the Bernoulli Distribution

6.1 A Powerful Trick

Definition: Indicator Random Variable

For an event A, the indicator $\mathbf{1}_A$ equals:

$$\mathbf{1}_A = \begin{cases} 1 \text{ if } \ A \ \text{ occurs} \\ 0 \text{ otherwise} \end{cases}$$

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Key property: $E(\mathbf{1}_A) = P(A)$

Proof: $E(\mathbf{1}_A) = 1 \cdot P(A) + 0 \cdot P(A^c) = P(A)$

6.2 The Bernoulli Distribution

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Definition: Bernoulli Distribution

$X \sim \text{Bernoulli}(p)$ if X takes value 1 with probability p and 0 with probability $1 - p$.

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

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Mean: $E(X) = p$

Variance: $\text{Var}(X) = p(1 - p)$

Examples:

- Coin flip: heads or tails
- Email: spam or not spam
- Patient: has disease or not

6.3 Example: Counting Heads

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Trick: Write X as a sum of indicators (Bernoulli RVs)!

$$X = \mathbf{1}_1 + \mathbf{1}_2 + \ldots + \mathbf{1}_{100}$$

where $\mathbf{1}_i \sim \text{Bernoulli}\left(\frac{1}{2}\right)$.

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$$E(X) = E(\mathbf{1}_1) + E(\mathbf{1}_2) + \ldots + E(\mathbf{1}_{100})$$

$$= P(\text{heads}) + P(\text{heads}) + \ldots = 100 \cdot \frac{1}{2} = 50$$

7. The Geometric Distribution

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Example: Rolling a Six

Roll a die until you get a 6. How many rolls on average?

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Definition: Geometric Distribution

$X \sim \text{Geometric}(p)$ counts the number of trials until the first success, where each trial succeeds with probability p.

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \ldots$$

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Interpretation: $(k - 1)$ failures, then 1 success.

7.3 Geometric: Mean and Variance

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Note: Intuition: If you succeed 1 in p of the time, you need about $\frac{1}{p}$ tries.

7.4 The Coupon Collector Problem

Adapted from Grimmett & Stirzaker

A cereal company puts one of n different toys in each box (equally likely).

Question: On average, how many boxes until you collect all n toys?

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This seems hard... but linearity makes it easy!

7.5 Coupon Collector: Solution

Let T_j = boxes needed to get the jth **new** toy (after having $j - 1$).

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$\boxed{\text{For } n = 50 \text{ toys: } E(T) \approx 225 \text{ boxes!}}$

7.6 Try It Yourself: Matching Problem

Try it yourself

Talk to your neighbor and try to solve this problem.

n people each put their hat in a box. Hats are randomly redistributed.

Let X = number of people who get their own hat back.

Find $E(X)$.

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On average, exactly 1 person gets their own hat — regardless of n!

8. Variance Properties (Review)

8.1 Recall: Variance

Recall:

$$\mathrm{Var}(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2$$

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$\sigma(aX) = |a|\ \sigma(X)$

Shift:

$\mathrm{Var}(X + b) = \mathrm{Var}(X)$

Adding a constant doesn't change spread!

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Combined: $\mathrm{Var}(aX + b) = a^2\ \mathrm{Var}(X)$

8.2 Why $\mathrm{Var}(X + b) = \mathrm{Var}(X)$?

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$$= E\left[\left(X + b - E(X) - b\right)^2\right]$$

8.2 Why Var(X + b) = Var(X)?

$$\mathrm{Var}(X+b) = E\big[(X+b-E(X+b))^2\big]$$

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$$= E\left[(X-E(X))^2\right] = \mathrm{Var}(X)$$

Intuition: Shifting all values by b moves the mean by b too.

The **distances** from the mean don't change!

9. Higher-Order Moments

9.1 Moments: A Generalization

Definition: nth Moment

The nth moment of X is $E(X^n)$

The nth **central** moment is $E[(X-\mu)^n]$ where $\mu = E(X)$

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- 1st moment: **Mean** μ
- 2nd central moment: **Variance** σ^2
- 3rd central moment: related to **Skewness**
- 4th central moment: related to **Kurtosis**

9.2 Skewness: Asymmetry

Definition: Skewness

$$\text{Skew}(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{E[(X-\mu)^3]}{\sigma^3}$$

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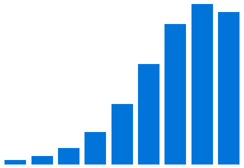
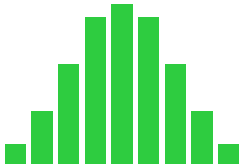
Negative skew



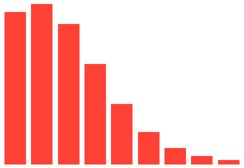
Left tail longer

Symmetric



Skew ≈ 0

Positive skew



Right tail longer

9.3 Kurtosis: Tail Heaviness

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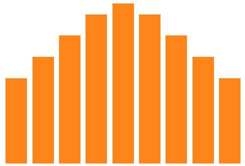
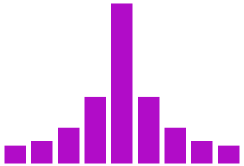
Low kurtosis (platykurtic)



Thin tails, flat peak

High kurtosis (leptokurtic)



Heavy tails, sharp peak

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Note: In practice, 3rd and 4th moments are most commonly used. Higher moments are rarely estimated reliably from data.

9.5 Recap

Today we covered:

- $E(aX + b) = aE(X) + b$; LOTUS: $E(g(X)) = \sum_x g(x)p(x)$
- Uniform$\{a, ..., b\}$: $E(X) = \frac{a+b}{2}$; die roll is Uniform$\{1, ..., 6\}$
- Bernoulli(p): $E(X) = p$; Geometric(p): $E(X) = 1/p$
- Indicators + linearity: powerful for counting problems
- Higher moments: skewness (asymmetry), kurtosis (tail weight)