0.1 Learning Objectives

By the end of this lecture, you will be able to:

- Understand the normal (Gaussian) distribution and its properties
- Recognize when to use the normal distribution
- Standardize data using z-scores
- Use the standard normal distribution to compute probabilities

0.3 The Normal Distribution

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Probability Density Function:

$$\mathrm{PDF}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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Variance (σ^2):

- Spread of the distribution
- Larger variance = wider bell curve
- $\mathrm{Var}(X) = \sigma^2$
- Standard deviation: $\sigma = \sqrt{\mathrm{Var}(X)}$

0.5 Visualizing the Normal Distribution

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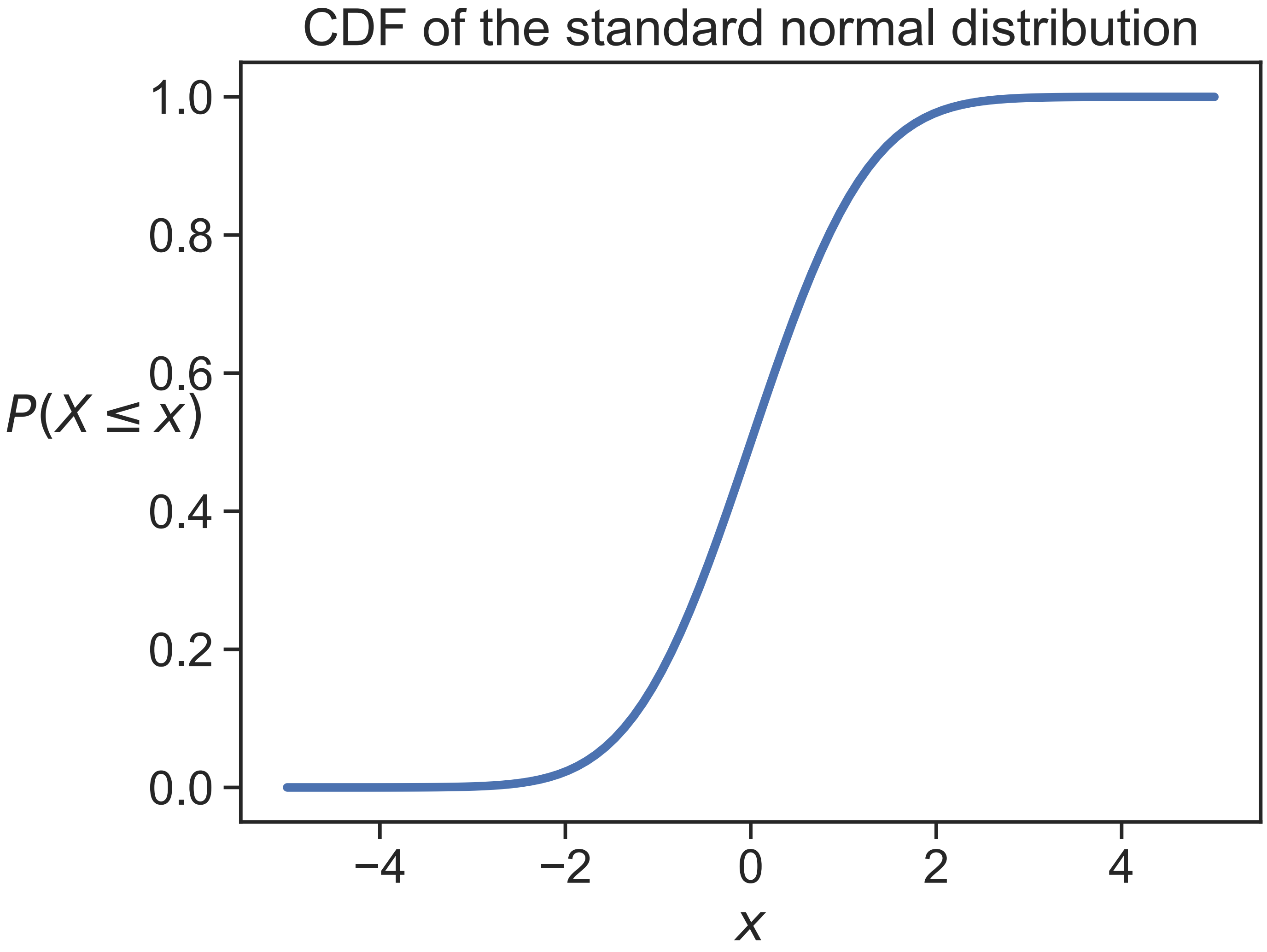

1. Visualizing Data Distributions

1.1 Median, Quantiles, and Quartiles

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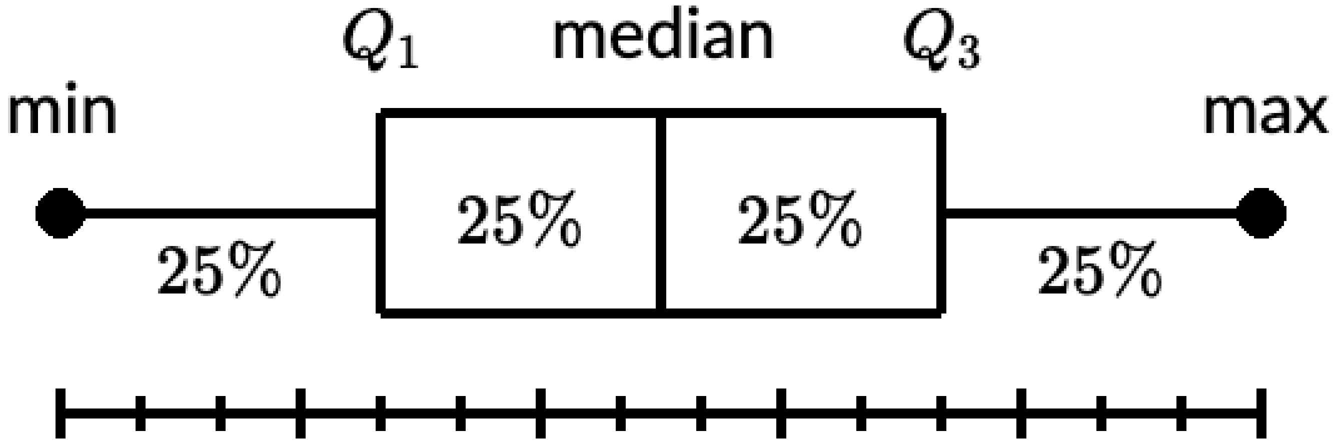
Quartile: 25% of the observations of X are less than or equal to the **first quartile**

- Second quartile = median
- Third quartile = 75th percentile = $Q_{75}(X)$
- **Inter-Quartile Range (IQR)** = $Q_{75}(X) - Q_{25}(X)$

1.2 Visualization: Boxplot



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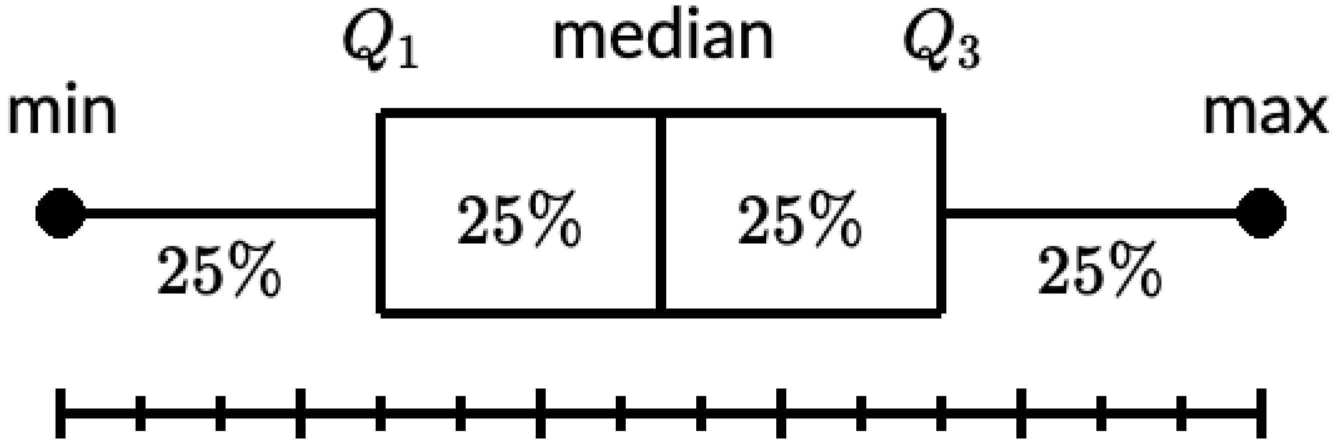

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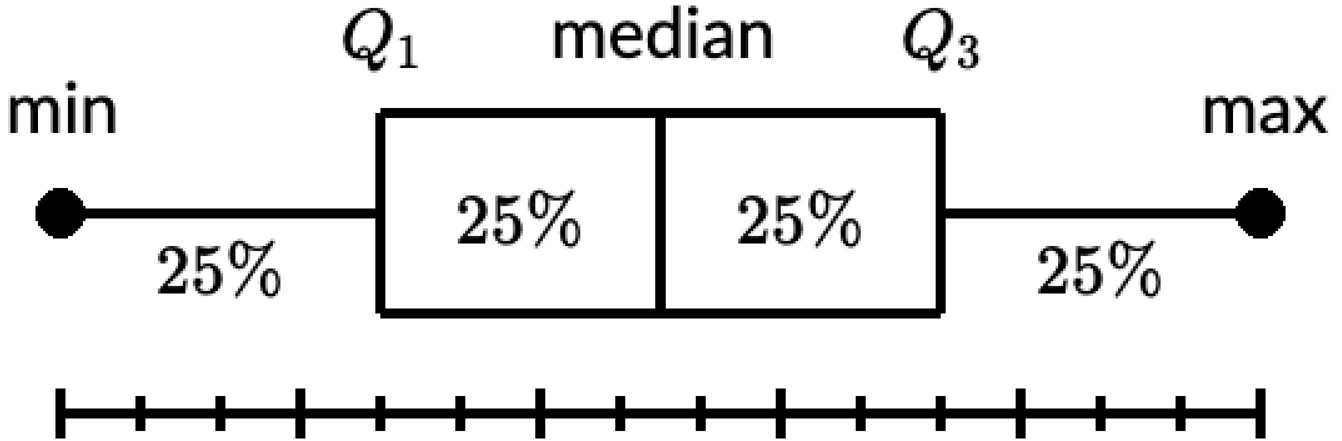

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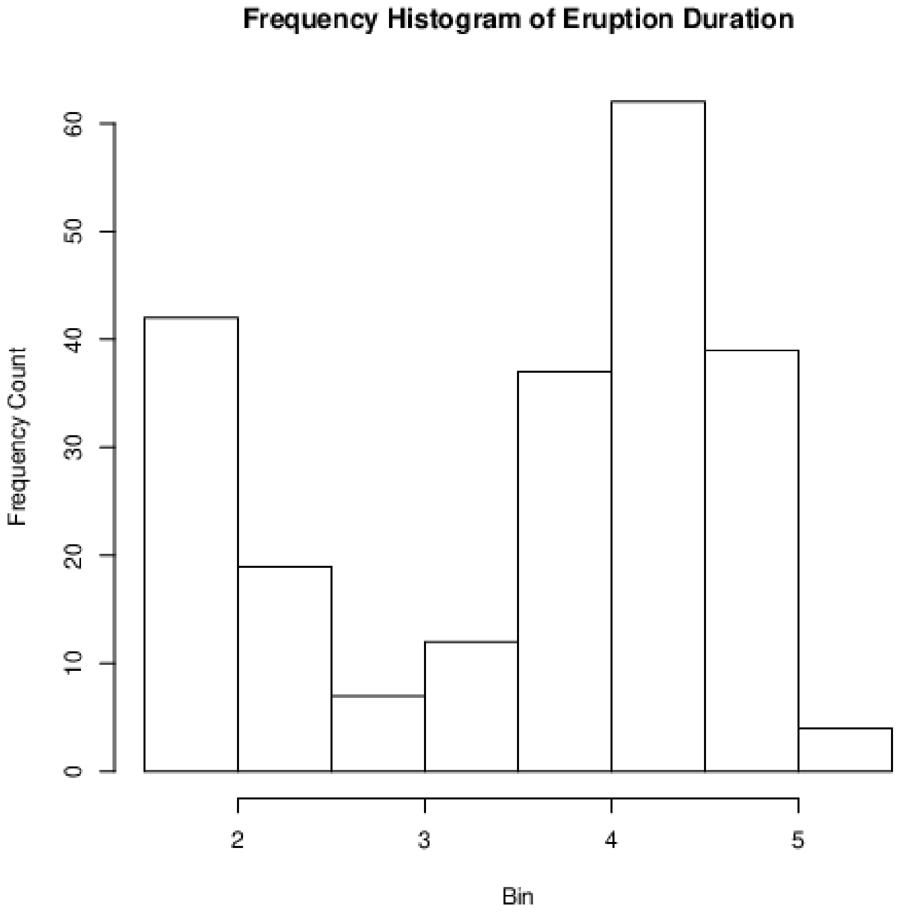
Boxplots show the distribution of data through quartiles and identify outliers.

1.3 Visualization: Histogram



A **histogram** displays the distribution of data by dividing it into bins and showing the frequency of observations in each bin.

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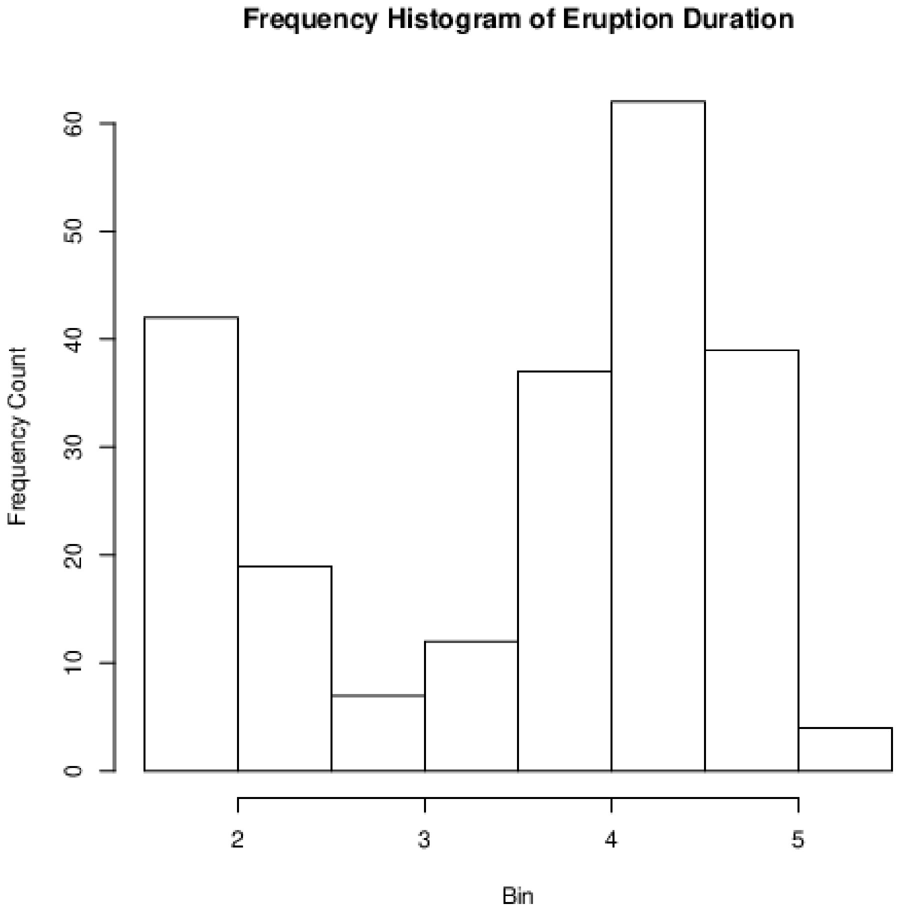
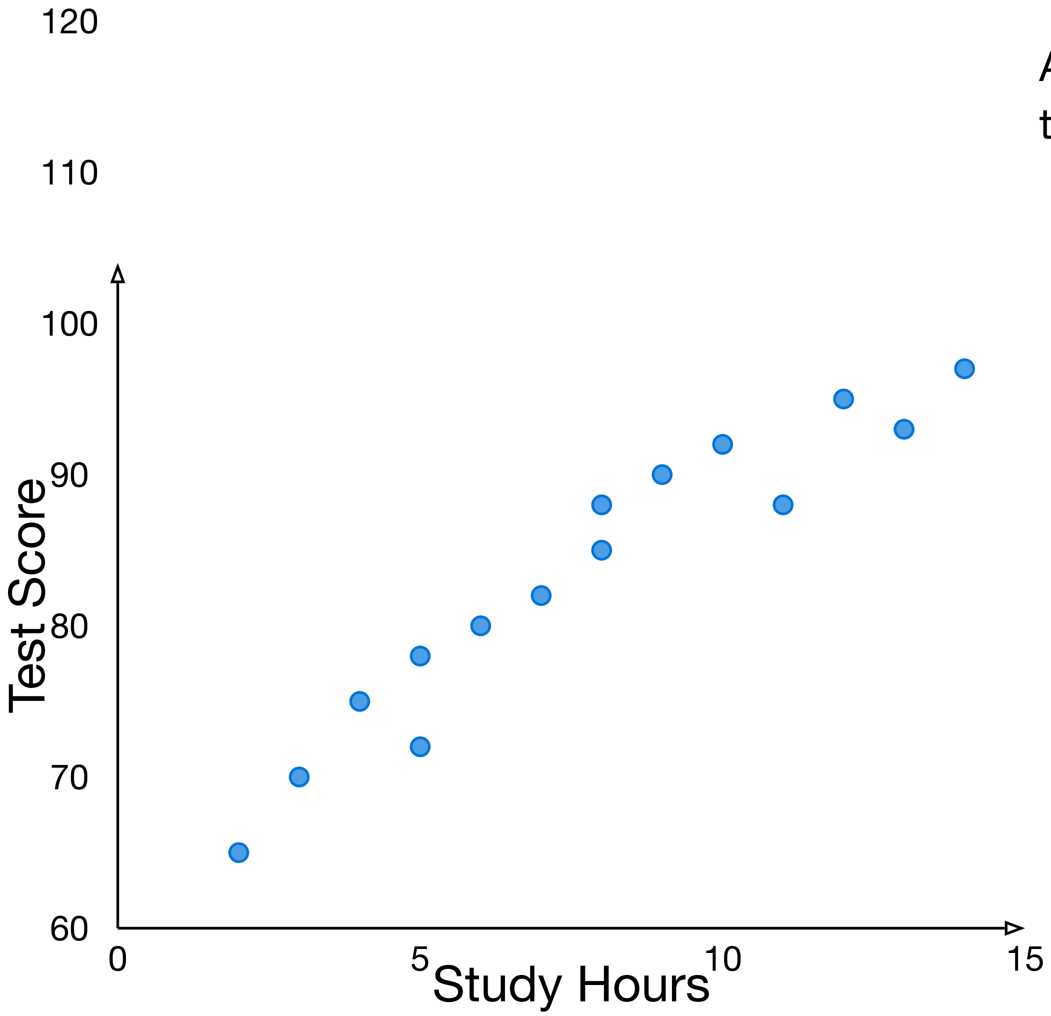


A **histogram** displays the distribution of data by dividing it into bins and showing the frequency of observations in each bin.

Key features:

- X-axis: range of values (bins)
- Y-axis: frequency or count
- Shape reveals the distribution pattern

1.4 Visualization: Scatter Plot



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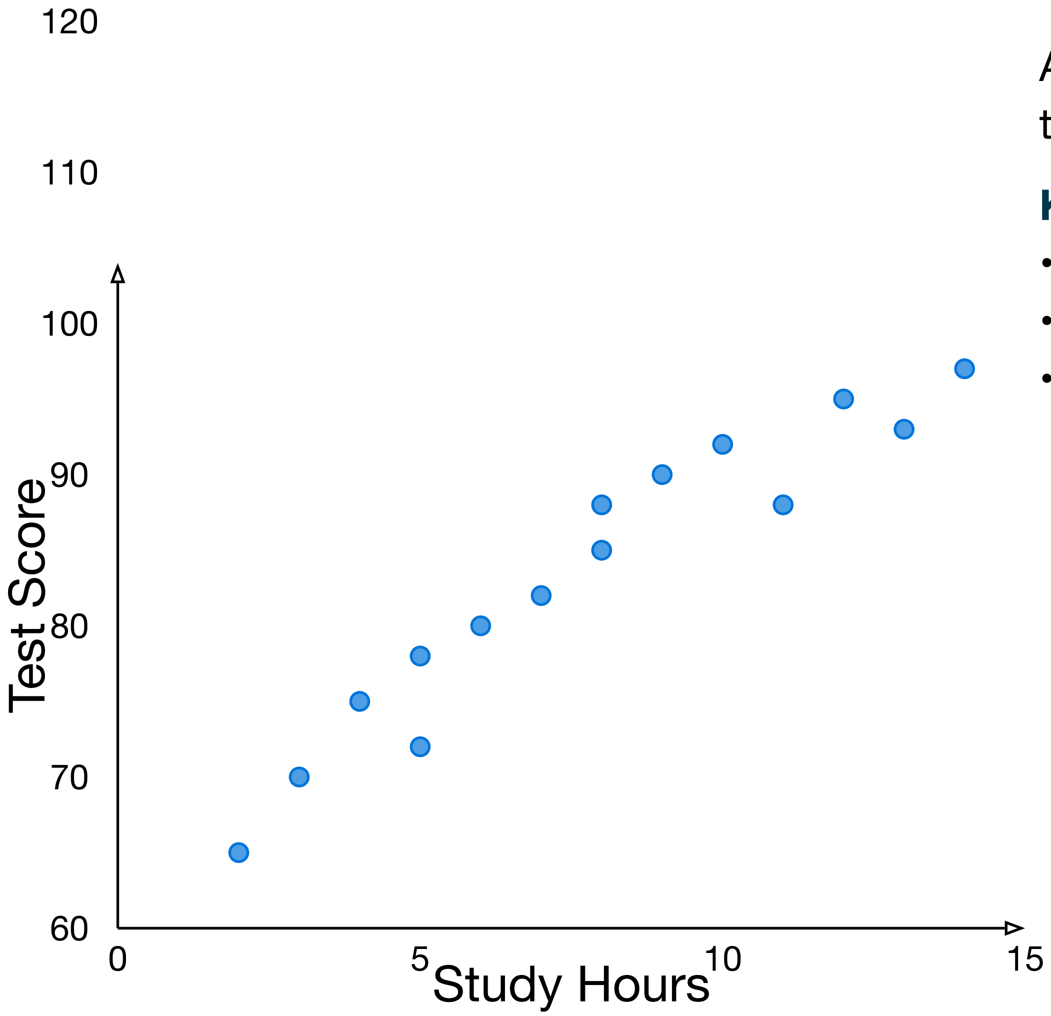

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- Each point represents one observation
- Shows correlation patterns
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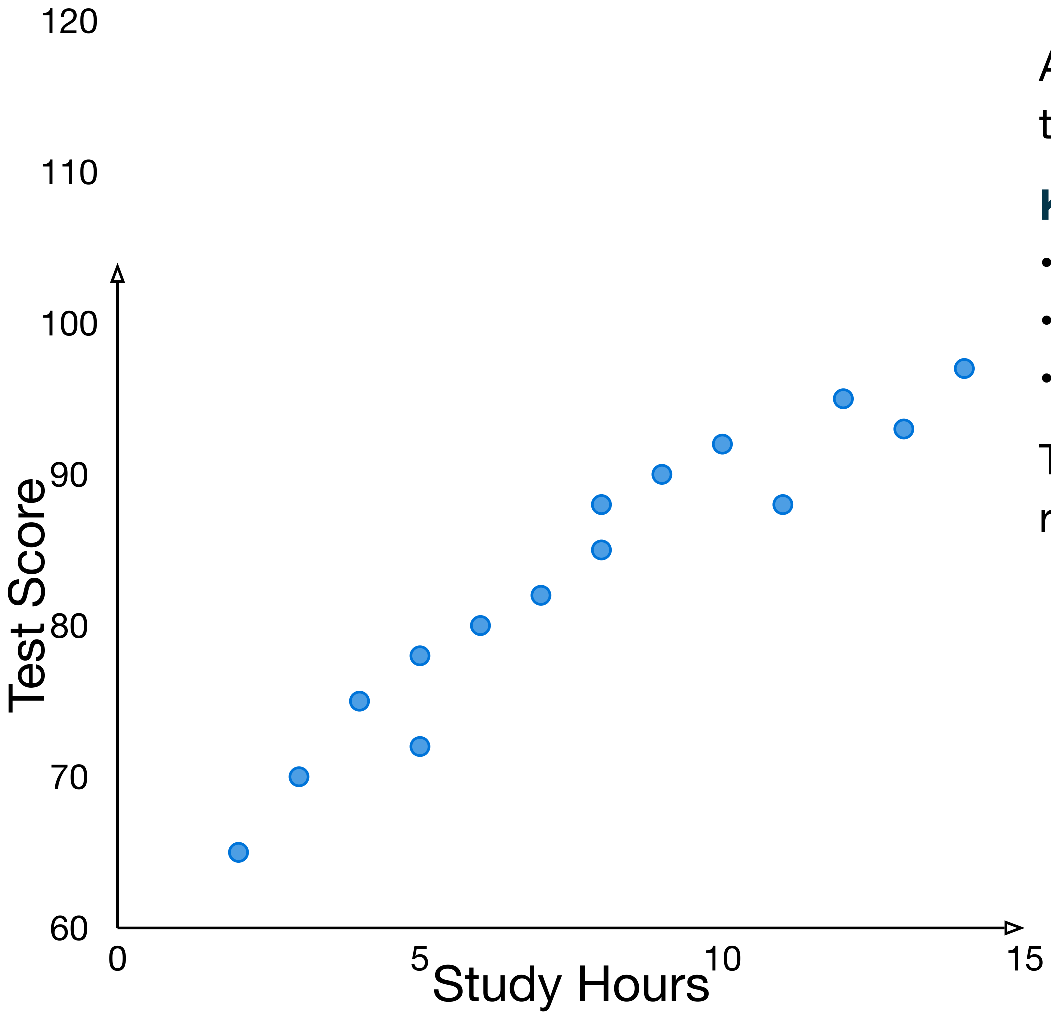

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Key features:

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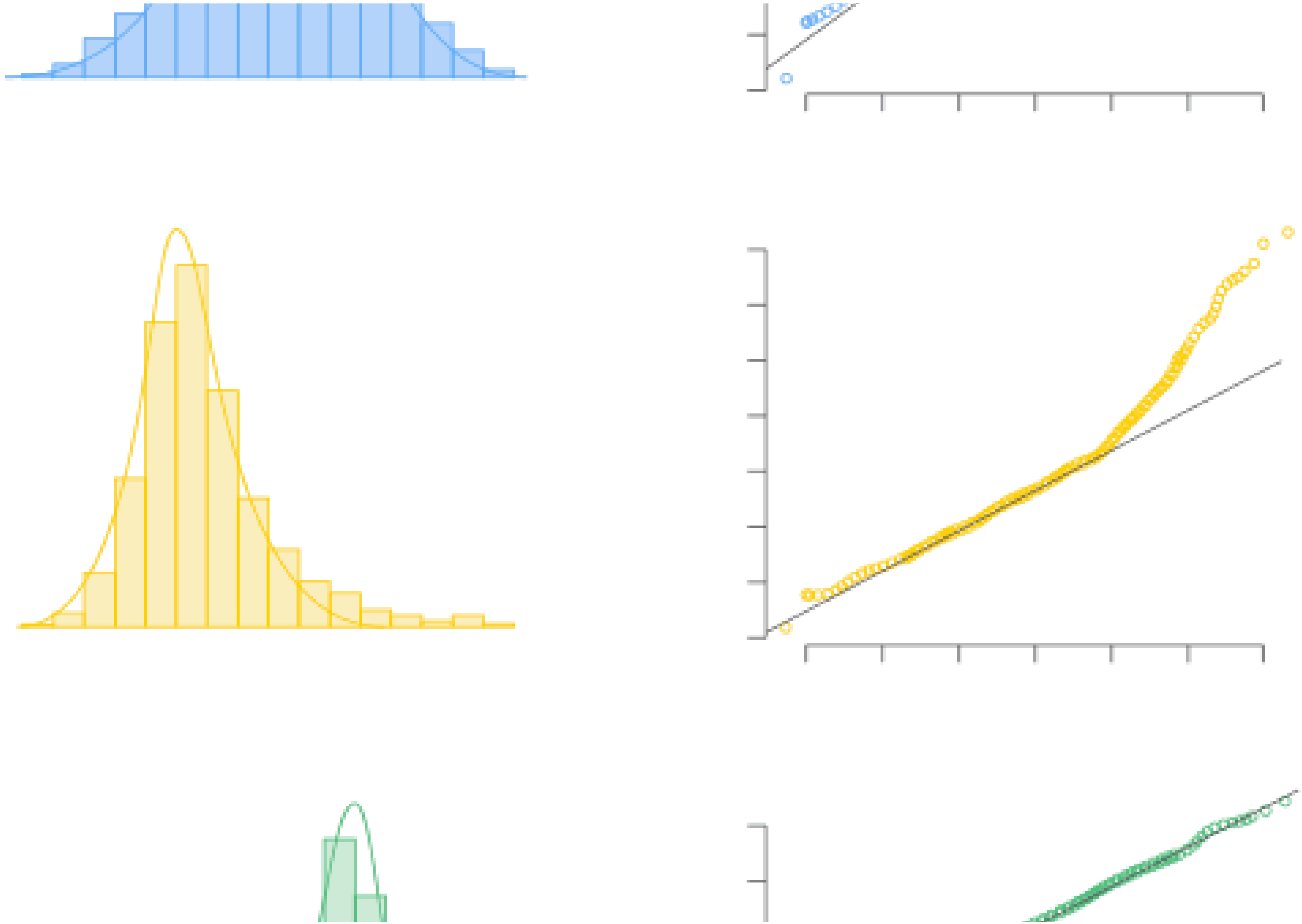
This example shows study hours vs. test scores, revealing a positive correlation.

1.5 Quantile-Quantile Plot

Right-skewed data



2. Back to the Normal Distribution

2.1 Why is the Normal Distribution Important?

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The sum of many independent and identically distributed (i.i.d.) random variables approaches a Normal distribution, regardless of the original distribution.

This means that many real-world phenomena that result from combining many small independent effects follow a normal distribution:

- Heights of people
- Measurement errors
- Test scores (under certain conditions)
- Stock market returns (approximately)

2.2 Example: Heights

Suppose adult male heights in the US follow a normal distribution with:

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What is the probability that a randomly selected adult male is between 67 and 73 inches tall?

2.3 Computing Probabilities: The Challenge

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Solutions:

- Use numerical methods (computers do this)
- Use standardization + lookup tables
- Use statistical software (R, Python, etc.)

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This is incredibly useful because:

- We only need tables/functions for ONE distribution ($\mathrm{Norm}(0, 1)$)
- All normal distribution probabilities can be computed using standardization

2.7 Z-Scores

The standardized value $Z = \frac{X-\mu}{\sigma}$ is called a **Z-score**.

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Interpretation:

- Z-score tells us how many standard deviations away from the mean a value is
- $Z = 0$: value equals the mean
- $Z = 1$: value is 1 standard deviation above the mean
- $Z = -2$: value is 2 standard deviations below the mean

2.8 Example: Computing a Z-Score

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What is the Z-score for a height of 76 inches?

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A height of 76 inches is 2 standard deviations above the mean.

2.9 Example: Computing Probabilities

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2.9 Example: Computing Probabilities

Adult male heights: $X \sim \mathrm{Norm}(70, 9)$

What is $P(X \leq 73)$?

Step 1: Standardize

$$Z = \frac{73 - 70}{3} = 1$$

Step 2: Use standard normal CDF

$$P(X \leq 73) = P(Z \leq 1) = \Phi(1)$$

2.10 The Z-Table

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Note: Today, we typically use software rather than tables, but understanding Z-scores is still essential.

2.11 The Standard Normal Table (Z-Table)

The Z-table shows $\Phi(z) = P(Z \leq z)$ for the standard normal distribution:

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	**0.8413**	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

For negative z: use symmetry $\Phi(-z) = 1 - \Phi(z)$