0.1 Learning Objectives

By the end of this lecture, you will be able to:

- Define stochastic processes and state spaces
- State the Markov property and explain its significance
- Construct transition matrices from verbal descriptions
- Compute n-step transition probabilities via matrix powers
- Find the steady-state distribution of a Markov chain

1. Stochastic Processes

1.1 What Is a Stochastic Process?

Definition: Stochastic Process

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- Can be **discrete** or **continuous** in time
- Can be **discrete** or **continuous** in state space
- Today: discrete time, discrete state space

1.2 States and State Spaces

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The **state space** S is the set of all possible states.

Example: Coin Flip Process

Flip a fair coin 100 times. Let X_t = outcome of flip t.

- State space: $S = \{H, T\}$
- Each X_t is a random variable taking values in S

1.3 Example: Random Walk

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- Tails: -1

Let X_t = score after t flips.

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$$S = \{-100, -99, ..., 99, 100\}$$

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This is a **random walk**

1.3 Example: Random Walk

one of the simplest stochastic processes.

1.4 Evolution of a Stochastic Process

If $X_1 = i$ and $X_2 = j$, we say the process made a **transition** from state i to state j.

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Note: In general, the future of a stochastic process can depend on its entire history. This makes analysis very hard. Can we simplify?

2. Markov Chains

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Question: What if the future only depends on the **present**, not the past?

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Definition: Markov Property

A stochastic process $\{X_1, X_2, X_3, ...\}$ has the **Markov property** if

$$P(X_{n+1} = j \mid X_n = i_n, X_{n-1} = i_{n-1}, ..., X_1 = i_1) = P(X_{n+1} = j \mid X_n = i_n)$$

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The next state depends *only* on the current state

2.1 The Key Simplification

not on how we got here.

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It captures **one-step dependence**: the future depends on the present, but not the past.

We are interested in the **long-run behavior**: once the process no longer "remembers" its starting state.

2.3 Example: Weather

Example: Island Weather

On any day, the weather is sunny (S) or rainy (R).

$$X_n = \begin{cases} 0 \text{ if sunny} \\ 1 \text{ if rainy} \end{cases}$$

Once a pattern begins, it tends to continue:

- If today is sunny, probability 0.97 that tomorrow is sunny
- If today is rainy, probability 0.94 that tomorrow is rainy

2.4 Weather: Transition Probabilities

Today	Tomorrow	Notation	Probability
Sunny	Sunny	$P(X_{n+1} = 0 \mid X_n = 0)$	0.97
Sunny	Rainy	$P(X_{n+1} = 1 \mid X_n = 0)$	0.03
Rainy	Sunny	$P(X_{n+1} = 0 \mid X_n = 1)$	0.06
Rainy	Rainy	$P(X_{n+1} = 1 \mid X_n = 1)$	0.94

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Notice: each pair of rows sums to 1 (from any state, we must go *somewhere*).

3. Transition Matrices

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Properties:

- All entries are between 0 and 1
- Each row sums to 1 (each row is a probability distribution)

3.2 Try It: Build a Transition Matrix

Example: Social Media

A user is on one of three pages: **Home** (H), **Profile** (P), or **Search** (S).

- From Home: 60% stay, 30% go to Profile, 10% go to Search
- From Profile: 40% go to Home, 40% stay, 20% go to Search
- From Search: 30% go to Home, 20% go to Profile, 50% stay

Write the transition matrix.

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- From Profile: 40% go to Home, 40% stay, 20% go to Search
- From Search: 30% go to Home, 20% go to Profile, 50% stay

Write the transition matrix.

Try it yourself

Talk to your neighbor and try to solve this problem.

3.3 Solution: Social Media Transition Matrix

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

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Reading row by row: $P_{HH} = 0.6$, $P_{HP} = 0.3$, $P_{HS} = 0.1$, etc.

Note: Each row sums to 1. We can think of each row as the conditional distribution of the next state, given the current state.

4. Multi-Step Transitions

4.1 n-Step Transition Probabilities

Definition: n-Step Transition Probability

The probability of going from state i to state j in exactly n steps:

$$P_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$$

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Key fact: The n-step transition matrix is P^n (the matrix P raised to the n-th power).

Row i of P^n gives the distribution of X_n given that the chain started in state i.

4.2 Try It: Two-Step Weather Transitions

Example: Weather After 2 Days

Using the weather transition matrix $P = \begin{pmatrix} 0.97 & 0.03 \\ 0.06 & 0.94 \end{pmatrix}$:

If today is sunny, what is the probability it's rainy in 2 days?

4.2 Try It: Two-Step Weather Transitions

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If today is sunny, what is the probability it's rainy in 2 days?

Try it yourself

Talk to your neighbor and try to solve this problem.

4.3 Solution: Two-Step Weather

$$P^2 = P \cdot P = \begin{pmatrix} 0.97 & 0.03 \\ 0.06 & 0.94 \end{pmatrix} \cdot \begin{pmatrix} 0.97 & 0.03 \\ 0.06 & 0.94 \end{pmatrix}$$

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$$= \begin{pmatrix} 0.97 \cdot 0.97 + 0.03 \cdot 0.06 & 0.97 \cdot 0.03 + 0.03 \cdot 0.94 \\ 0.06 \cdot 0.97 + 0.94 \cdot 0.06 & 0.06 \cdot 0.03 + 0.94 \cdot 0.94 \end{pmatrix}$$

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$$P_{0,1}^{(2)} = 0.0573$$

4.3 Solution: Two-Step Weather

about a 5.7% chance of rain in 2 days given sun today.

5. Steady-State Distribution

5.1 Long-Run Behavior

What happens as $n \to \infty$? Do the rows of P^n converge?

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Example: Weather: Powers of P

$$P^1 = \begin{pmatrix} 0.97 & 0.03 \\ 0.06 & 0.94 \end{pmatrix} \qquad P^{10} \approx \begin{pmatrix} 0.76 & 0.24 \\ 0.48 & 0.52 \end{pmatrix}$$

$$P^{50} \approx \begin{pmatrix} 0.667 & 0.333 \\ 0.667 & 0.333 \end{pmatrix} \qquad P^{100} \approx \begin{pmatrix} 0.667 & 0.333 \\ 0.667 & 0.333 \end{pmatrix}$$

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The rows converge to the **same** distribution! The starting state no longer matters.

5.2 Steady-State Distribution

Definition: Steady-State (Stationary) Distribution

A probability distribution π is a **steady-state distribution** if it satisfies:

$$\pi P = \pi$$

That is, π is unchanged by the transition — the system is in equilibrium.

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Recall: An eigenvector of a matrix A satisfies $Av = \lambda v$. Here we need $\pi P = 1 \cdot \pi$.

5.3 Finding the Steady-State Distribution

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Three approaches:

1. **Simulate**: run the chain for many steps, record the fraction of time in each state
2. **Matrix powers**: compute P^n for large n and read off any row
3. **Solve directly**: solve $\pi P = \pi$ subject to $\sum_i \pi_i = 1$

5.4 Try It: Weather Steady-State

Example: Weather Steady-State

Find $\pi = (\pi_0, \pi_1)$ such that $\pi P = \pi$ and $\pi_0 + \pi_1 = 1$.

$$(\pi_0, \pi_1)\begin{pmatrix} 0.97 & 0.03 \\ 0.06 & 0.94 \end{pmatrix} = (\pi_0, \pi_1)$$

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5.5 Solution: Weather Steady-State

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$$\pi = (2/3, 1/3) = \boxed{(0.667, 0.333)}$$

In the long run, about 67% of days are sunny and 33% are rainy

5.5 Solution: Weather Steady-State

regardless of today's weather.

6. Properties of Markov Chains

6.1 When Does Convergence Happen?

Not all Markov chains converge to a steady state. Two conditions guarantee it:

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Definition: Aperiodicity

A state i has **period** d if returns to i must occur in multiples of d steps. A chain is **aperiodic** if all states have period 1.

6.2 Convergence Theorem

If a Markov chain is **irreducible** and **aperiodic**, then:

$$P_{ij}^n \to \pi_j \text{ as } n \to \infty$$

for all states i and j.

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Furthermore, π is **unique** and satisfies:

- $\pi P = \pi$ (stationarity)
- $\sum_i \pi_i = 1$ (valid probability distribution)
- π_i = expected long-run fraction of time spent in state i

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Note: These properties make Markov chains a powerful modeling tool. Many real-world processes — web browsing, weather, queues, genetics — are well-approximated by Markov chains.

6.4 Recap

Today we covered:

- A **stochastic process** is a collection of random variables indexed by time
- The **Markov property**: the future depends only on the present, not the past
- **Transition matrices** encode one-step probabilities; P^n gives n-step probabilities
- The **steady-state distribution** π satisfies $\pi P = \pi$ — it's where the chain "settles"
- Irreducible + aperiodic chains converge to a unique steady state