0.1 Learning Objectives

By the end of this lecture, you will be able to:

- Construct and interpret confidence intervals for population means
- Distinguish z-intervals from t-intervals
- Understand the connection between CIs and hypothesis tests

0.2 Study Buddy Matching from AggieWorks

1. From Point Estimates to Intervals

1.1 The Problem with Point Estimates

A polling firm surveys $n = 1200$ likely voters. They find 53% support Candidate A.

1.1 The Problem with Point Estimates

A polling firm surveys $n = 1200$ likely voters. They find 53% support Candidate A.

Should we report: “Candidate A will win with 53% of the vote”?

1.1 The Problem with Point Estimates

A polling firm surveys $n = 1200$ likely voters. They find 53% support Candidate A.

Should we report: “Candidate A will win with 53% of the vote”?

No

1.1 The Problem with Point Estimates

we know the sample proportion $\hat{p} = 0.53$ is just an *estimate*. A different sample of 1200 voters would give a different number.

1.1 The Problem with Point Estimates

we know the sample proportion $\hat{p} = 0.53$ is just an *estimate*. A different sample of 1200 voters would give a different number.

What we really want: a **range** that probably contains the true proportion.

1.2 "Margin of Error"

News reports say: "53% support Candidate A, **with a margin of error of $\pm$ 3%**."

1.2 “Margin of Error”

News reports say: “53% support Candidate A, **with a margin of error of $\pm$ 3%**.”

This means: we are fairly confident the true support is between 50% and 56%.

1.2 “Margin of Error”

News reports say: “53% support Candidate A, **with a margin of error of $\pm$ 3%**.”

This means: we are fairly confident the true support is between 50% and 56%.

Note: The “margin of error” is a confidence interval in disguise. Today we’ll learn exactly where it comes from.

1.3 Point Estimate vs. Interval Estimate

Point estimate

$\hat{\mu}$ popln. parameter

A single number: $\hat{p} = 0.53$

Precise but gives no sense of uncertainty.

Interval estimate

A range: $(0.50, 0.56)$

Less precise but communicates how much we trust the estimate.

1.3 Point Estimate vs. Interval Estimate

Point estimate

A single number: $\hat{p} = 0.53$

Precise but gives no sense of uncertainty.

Interval estimate

A range: $(0.50, 0.56)$

Less precise but communicates how much we trust the estimate.

Confidence intervals give us a principled way to construct interval estimates.

2. Constructing Confidence Intervals

2.1 Starting from the CLT

Recall: By the Central Limit Theorem, for a large i.i.d. sample $X_1, ..., X_n$ with mean μ and variance σ^2:

$$\overline{X} \cdot \sim \text{Normal}\left(\mu, \frac{\sigma^2}{n}\right)$$

2.1 Starting from the CLT

Recall: By the Central Limit Theorem, for a large i.i.d. sample $X_1, ..., X_n$ with mean μ and variance σ^2:

$$\overline{X} \cdot \sim \text{Normal}\left(\mu, \frac{\sigma^2}{n}\right)$$

Standardizing:

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \cdot \sim \text{Normal}(0, 1)$$

2.1 Starting from the CLT

Recall: By the Central Limit Theorem, for a large i.i.d. sample $X_1, ..., X_n$ with mean μ and variance σ^2:

$$\overline{X} \cdot\sim \text{Normal}\left(\mu, \frac{\sigma^2}{n}\right)$$

Standardizing:

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \cdot\sim \text{Normal}(0, 1)$$

From the standard normal table: $P(-1.96 < Z < 1.96) = 0.95$

2.2 Deriving the Confidence Interval

Starting from:

$$P\left(-1.96 < \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.96\right) = 0.95$$

2.2 Deriving the Confidence Interval

Starting from:

$$P\left(-1.96 < \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.96\right) = 0.95$$

Multiply through by $\frac{\sigma}{\sqrt{n}}$:

$$P\left(-1.96 \cdot \frac{\sigma}{\sqrt{n}} < \overline{X} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0.95$$

2.2 Deriving the Confidence Interval

Starting from:

$$P\left(-1.96 < \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < 1.96\right) = 0.95$$

Multiply through by $\frac{\sigma}{\sqrt{n}}$:

$$P\left(-1.96 \cdot \frac{\sigma}{\sqrt{n}} < \overline{X}-\mu < 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0.95$$

Rearrange to isolate μ:

$$P\left(\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right) = 0.95$$

2.3 The 95% Confidence Interval

Definition: 95% Confidence Interval for the Mean

$$\overline{x} \pm 1.96 \cdot \frac{s}{\sqrt{n}}$$

where $\overline{x}$ is the sample mean, s is the sample standard deviation, and n is the sample size.

2.3 The 95% Confidence Interval

Definition: 95% Confidence Interval for the Mean

$$\overline{x} \pm 1.96 \cdot \frac{s}{\sqrt{n}}$$

where $\overline{x}$ is the sample mean, s is the sample standard deviation, and n is the sample size.

The **margin of error** is:

$$\mathrm{ME} = 1.96 \cdot \frac{s}{\sqrt{n}}$$

2.3 The 95% Confidence Interval

Definition: 95% Confidence Interval for the Mean

$$\overline{x} \pm 1.96 \cdot \frac{s}{\sqrt{n}}$$

where $\overline{x}$ is the sample mean, s is the sample standard deviation, and n is the sample size.

The **margin of error** is:

$$\mathrm{ME} = 1.96 \cdot \frac{s}{\sqrt{n}}$$

This is the “margin of error” from polling reports

2.3 The 95% Confidence Interval

now you know where it comes from.

2.4 Try It: Quick Calculation

A survey of $n = 1200$ voters finds $\hat{p} = 0.53$. For proportions, $s \approx \sqrt{\hat{p}(1 - \hat{p})}$.

What is the margin of error?

2.4 Try It: Quick Calculation

A survey of $n = 1200$ voters finds $\hat{p} = 0.53$. For proportions, $s \approx \sqrt{\hat{p}(1-\hat{p})}$.

What is the margin of error?

Think (30 seconds): Compute $1.96 \cdot \frac{\sqrt{0.53 \cdot 0.47}}{\sqrt{1200}}$.

2.4 Try It: Quick Calculation

A survey of $n = 1200$ voters finds $\hat{p} = 0.53$. For proportions, $s \approx \sqrt{\hat{p}(1 - \hat{p})}$.

What is the margin of error?

Think (30 seconds): Compute $1.96 \cdot \frac{\sqrt{0.53 \cdot 0.47}}{\sqrt{1200}}$.

Try it yourself

Talk to your neighbor and try to solve this problem.

2.5 Solution

$$\text{ME} = 1.96 \cdot \frac{\sqrt{0.53 \times 0.47}}{\sqrt{1200}} = 1.96 \cdot \frac{\sqrt{0.2491}}{34.64} = 1.96 \cdot 0.0144 \approx 0.028$$

2.5 Solution

$$\mathrm{ME} = 1.96 \cdot \frac{\sqrt{0.53 \times 0.47}}{\sqrt{1200}} = 1.96 \cdot \frac{\sqrt{0.2491}}{34.64} = 1.96 \cdot 0.0144 \approx 0.028$$

So the margin of error is about **2.8%**

2.5 Solution

roughly the "$\pm 3\%$" you see in news reports.

2.5 Solution

roughly the "$\pm 3\%$" you see in news reports.

95% CI: $(0.53 - 0.028, 0.53 + 0.028) = (0.502, 0.558)$

2.6 Other Confidence Levels

The 1.96 comes from the standard normal distribution. For other confidence levels:

Confidence Level	α	$z_{\alpha/2}$
90%	0.10	1.645
95%	0.05	1.96
99%	0.01	2.576

2.6 Other Confidence Levels

The 1.96 comes from the standard normal distribution. For other confidence levels:

Confidence Level	α	$z_{\alpha/2}$
90%	0.10	1.645
95%	0.05	1.96
99%	0.01	2.576

Higher confidence $\rightarrow$ wider interval. There is a trade-off between confidence and precision.

3. z-Intervals vs. t-Intervals

3.1 When Is the Normal Approximation Good Enough?

The z-interval $\overline{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$ relies on the CLT.

3.1 When Is the Normal Approximation Good Enough?

The z-interval $\overline{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$ relies on the CLT.

This works well when n is large (say $n \geq 30$).

3.1 When Is the Normal Approximation Good Enough?

The z-interval $\overline{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$ relies on the CLT.

This works well when n is large (say $n \geq 30$).

But what about small samples?

3.1 When Is the Normal Approximation Good Enough?

The z-interval $\overline{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$ relies on the CLT.

This works well when n is large (say $n \geq 30$).

But what about small samples?

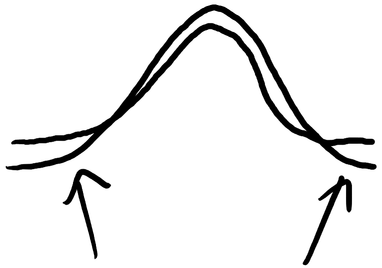
Note: When n is small, the sample standard deviation s is itself uncertain. We need to account for this extra uncertainty.

3.2 The t-Interval

Recall: The Student's t-distribution (from Lecture 9-1) has fatter tails than the normal — more probability in the extremes.

CLT: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $n > 30$

$\bar{X} \sim t$ $n < 30$



3.2 The t-Interval

Recall: The Student's t-distribution (from Lecture 9-1) has fatter tails than the normal — more probability in the extremes.

Definition: t-Interval for the Mean

$$\overline{x} \pm t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ is the critical value from the t-distribution with $n-1$ degrees of freedom.

3.2 The t-Interval

Recall: The Student's t-distribution (from Lecture 9-1) has fatter tails than the normal — more probability in the extremes.

Definition: t-Interval for the Mean

$$\overline{x} \pm t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ is the critical value from the t-distribution with $n-1$ degrees of freedom.

For large n, $t_{\alpha/2,n-1} \approx z_{\alpha/2}$ and the two intervals are nearly identical.

3.3 z vs. t: When to Use Which

	z-interval	t-interval
When	Large n (≥ 30)	Small n (< 30)
Uses	$z_{\alpha/2}$ from Normal	$t_{\alpha/2,n-1}$ from t-dist.
Critical value (95%, n=10)	1.96	2.26
Width	Narrower	Wider

3.3 z vs. t: When to Use Which

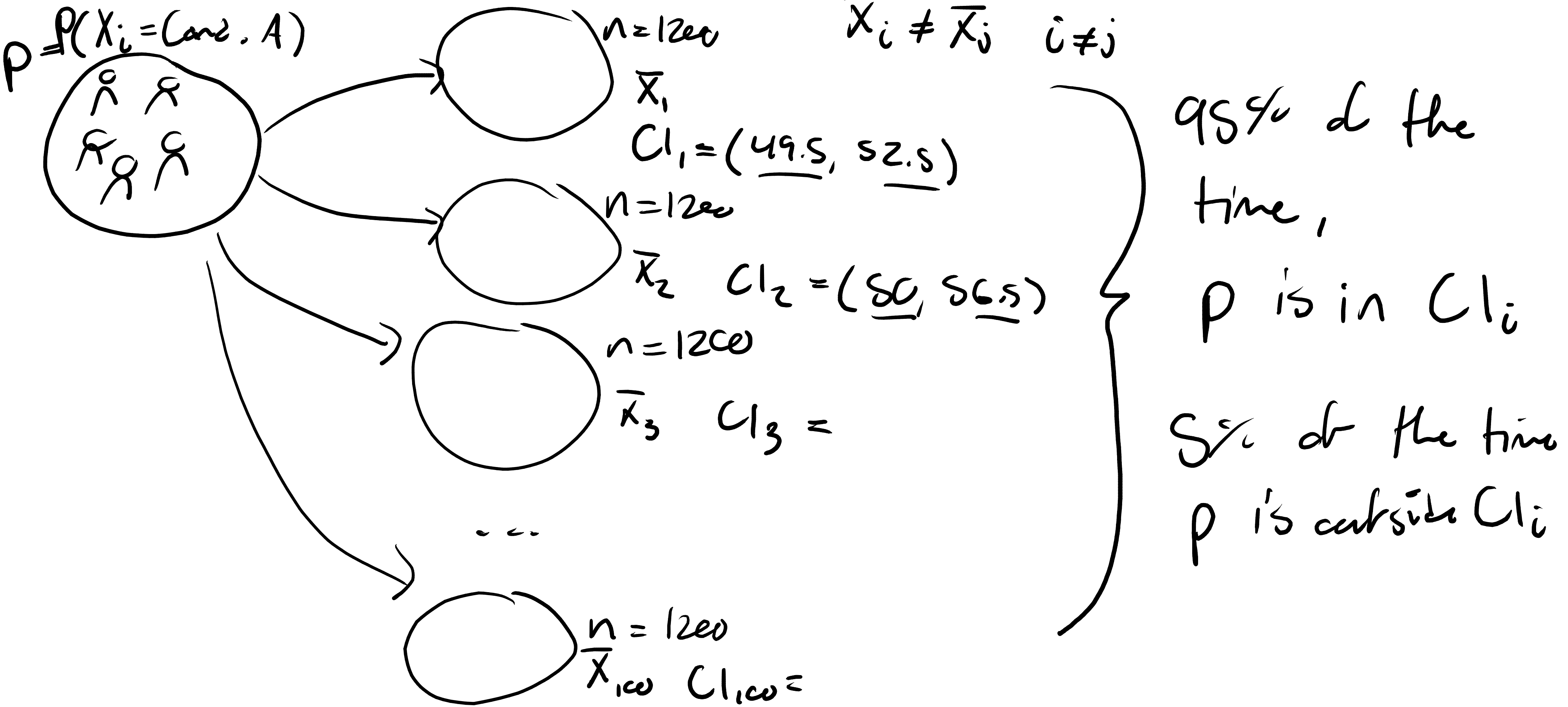
	z-interval	**t-interval**
When	Large n (≥ 30)	Small n (< 30)
Uses	$z_{\alpha/2}$ from Normal	$t_{\alpha/2,n-1}$ from t-dist.
Critical value (95%, n=10)	1.96	2.26
Width	Narrower	Wider

Note: In practice, many statisticians always use the t-interval. It's correct for any n, and for large n it gives the same answer as the z-interval.

4. Interpreting Confidence Intervals

4.1 What Does "95% Confident" Mean?

Imagine 100 different researchers each independently sample n observations and construct a 95% CI.



4.1 What Does "95% Confident" Mean?

Imagine 100 different researchers each independently sample n observations and construct a 95% CI.

About **95 of those 100 intervals** will contain the true parameter μ. ρ

About **5 will miss.**

4.1 What Does “95% Confident” Mean?

Imagine 100 different researchers each independently sample n observations and construct a 95% CI.

About **95 of those 100 intervals** will contain the true parameter μ.

About **5 will miss.**

“95% confidence” refers to the *procedure*, not to any single interval.

4.1 What Does “95% Confident” Mean?

Imagine 100 different researchers each independently sample n observations and construct a 95% CI.

About **95 of those 100 intervals** will contain the true parameter μ.

About **5 will miss.**

“95% confidence” refers to the *procedure*, not to any single interval.

Note: The true mean μ is a fixed number — it’s either in your interval or it isn’t. The randomness is in the *interval*, which changes with each sample.

4.2 A Common Misconception

Incorrect:

"There is a 95% probability that μ is in $(22.1, 28.7)$."

(μ is fixed, not random.)

Correct:

"We are 95% confident that $(22.1, 28.7)$ contains μ."

(The interval is random, constructed by a procedure that captures μ 95% of the time.)

4.3 CIs and Hypothesis Tests Are Two Sides of the Same Coin

Example: Espresso Machine (from Lecture 9-2)

We had $n = 12$, $\overline{x} = 25.4$, $s = 0.968$, testing $H_0 : \mu = 25$.

The 95% CI is:

$$25.4 \pm 2.20 \cdot \frac{0.968}{\sqrt{12}} = 25.4 \pm 0.615 = (24.79, 26.02)$$

Does this interval contain $\mu_0 = 25$? **Yes** — so we fail to reject H_0.

4.3 CIs and Hypothesis Tests Are Two Sides of the Same Coin

Example: Espresso Machine (from Lecture 9-2)

We had $n = 12$, $\overline{x} = 25.4$, $s = 0.968$, testing $H_0 : \mu = 25$.

The 95% CI is:

$$25.4 \pm 2.20 \cdot \frac{0.968}{\sqrt{12}} = 25.4 \pm 0.615 = (24.79, 26.02)$$

Does this interval contain $\mu_0 = 25$? **Yes** — so we fail to reject H_0.

1 - .95

Note: If the 95% CI contains μ_0, you fail to reject H_0 at $\alpha = 0.05$. If it doesn't, you reject. They always agree.

4.4 Why CIs Are Often More Useful

Hypothesis test: "Is the mean different from 25?" $\rightarrow$ Yes or No.

4.4 Why CIs Are Often More Useful

Hypothesis test: "Is the mean different from 25?" $\rightarrow$ Yes or No.

Confidence interval: "What values of μ are plausible?" $\rightarrow$ $(24.79, 26.02)$.

4.4 Why CIs Are Often More Useful

Hypothesis test: "Is the mean different from 25?" $\rightarrow$ Yes or No.

Confidence interval: "What values of μ are plausible?" $\rightarrow (24.79, 26.02)$.

The CI tells you *how different* the mean might be

4.4 Why CIs Are Often More Useful

not just *whether* it's different.

4.4 Why CIs Are Often More Useful

not just *whether* it's different.

Example: Comparing Two Drugs

$\alpha = 0.05$

- "Drug A is significantly better than Drug B ($p = 0.03$)." How much better?
- "Drug A reduces symptoms by 2–8 days more than Drug B (95% CI)." Much more informative.

↑ 95% CI

5. Worked Example

5.1 Computing a CI in R and Python

A sample of $n = 20$ UC Davis students sleep an average of $\overline{x} = 6.8$ hours per night with $s = 1.2$.

Construct a 95% confidence interval for the true mean.

5.1 Computing a CI in R and Python

A sample of $n = 20$ UC Davis students sleep an average of $\bar{x} = 6.8$ hours per night with $s = 1.2$.

Construct a 95% confidence interval for the true mean.

Since $n = 20 < 30$, use the t-interval. We need $t_{0.025,19}$:

quantile for t-dist

R

```r
qt(0.975, df=19)    # returns 2.093
```

Python

```python
scipy.stats.t.ppf(0.975, df=19)  # returns 2.093
```

5.1 Computing a CI in R and Python

A sample of $n = 20$ UC Davis students sleep an average of $\overline{x} = 6.8$ hours per night with $s = 1.2$.

Construct a 95% confidence interval for the true mean.

Since $n = 20 < 30$, use the t-interval. We need $t_{0.025,19}$:

```r
qt(0.975, df=19)    # returns 2.093
```
Vim

```python
scipy.stats.t.ppf(0.975, df=19)  # returns 2.093
```
Python

$$6.8 \pm 2.093 \cdot \frac{1.2}{\sqrt{20}} = 6.8 \pm 0.562 = (6.24, 7.36)$$

5.2 Try It: Construct a CI

A food truck sells burritos. Over $n = 50$ randomly selected orders, the mean preparation time is $\overline{x} = 4.2$ minutes with $s = 0.9$ minutes.

Construct a 95% confidence interval for the mean preparation time.

5.2 Try It: Construct a CI

A food truck sells burritos. Over $n = 50$ randomly selected orders, the mean preparation time is $\overline{x} = 4.2$ minutes with $s = 0.9$ minutes.

Construct a 95% confidence interval for the mean preparation time.

Think (1 minute): Should you use a z-interval or t-interval? Compute the CI.

5.2 Try It: Construct a CI

A food truck sells burritos. Over $n = 50$ randomly selected orders, the mean preparation time is $\overline{x} = 4.2$ minutes with $s = 0.9$ minutes.

Construct a 95% confidence interval for the mean preparation time.

Think (1 minute): Should you use a z-interval or t-interval? Compute the CI.

Discuss with your neighbor (1 minute): Compare your answers.

5.2 Try It: Construct a CI

A food truck sells burritos. Over $n = 50$ randomly selected orders, the mean preparation time is $\overline{x} = 4.2$ minutes with $s = 0.9$ minutes.

Construct a 95% confidence interval for the mean preparation time.

Think (1 minute): Should you use a z-interval or t-interval? Compute the CI.

Discuss with your neighbor (1 minute): Compare your answers.

Try it yourself

Talk to your neighbor and try to solve this problem.

5.3 Solution

$n = 50 \geq 30$, so a z-interval is appropriate (t-interval also fine).

$$4.2 \pm 1.96 \cdot \frac{0.9}{\sqrt{50}} = 4.2 \pm 1.96 \cdot 0.127 = 4.2 \pm 0.249$$

5.3 Solution

$n = 50 \geq 30$, so a z-interval is appropriate (t-interval also fine).

$$4.2 \pm 1.96 \cdot \frac{0.9}{\sqrt{50}} = 4.2 \pm 1.96 \cdot 0.127 = 4.2 \pm 0.249$$

95% CI: $(3.95, 4.45)$ minutes.

5.3 Solution

$n = 50 \geq 30$, so a z-interval is appropriate (t-interval also fine).

$$4.2 \pm 1.96 \cdot \frac{0.9}{\sqrt{50}} = 4.2 \pm 1.96 \cdot 0.127 = 4.2 \pm 0.249$$

95% CI: $(3.95, 4.45)$ minutes.

We are 95% confident that the true mean preparation time is between 3.95 and 4.45 minutes.

5.4 Recap

Today we covered:

- **Confidence intervals** give a range of plausible values for a parameter
- 95% CI for the mean: $\overline{x} \pm z_{\alpha/2} \cdot s/\sqrt{n}$ (large n) or $\overline{x} \pm t_{\alpha/2,n-1} \cdot s/\sqrt{n}$ (small n)
- **Interpretation:** 95% of intervals constructed this way contain the true μ
- **CI and hypothesis tests agree:** if μ_0 is outside the CI, reject H_0
- CIs are often more informative than p-values alone